

Criticality theory of half-linear equations with the (p, A) -Laplacian

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Abstract

We study positive solutions of half-linear second-order elliptic equations of the form

$$Q_{A,V}(u) := -\operatorname{div}(|\nabla u|_A^{p-2} A(x) \nabla u) + V(x)|u|^{p-2}u = 0 \quad \text{in } \Omega,$$

where $1 < p < \infty$, Ω is a domain in \mathbb{R}^n , $n \geq 2$, $V \in L^\infty_{\text{loc}}(\Omega)$, $A = (a_{ij}) \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^{n^2})$ is a symmetric and locally uniformly positive definite matrix in Ω , and

$$|\xi|_A^2 := \langle A(x)\xi, \xi \rangle = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \quad x \in \Omega, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

We extend criticality theory which has been established for linear operators and for half-linear operators involving the p -Laplacian, to the operator $Q_{A,V}$. We prove Liouville-type theorems, and study the behavior of positive solutions of the equation $Q_{A,V}(u) = 0$ near an isolated singularity and near infinity in Ω , and obtain some perturbations results.

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1. Introduction

In this paper we study positive solutions of half-linear elliptic partial differential equations of second-order. Recall that a partial differential equation $Q(u) = 0$ is said to be *half-linear* if for any $\alpha \in \mathbb{R}$ we have $Q(\alpha v) = 0$, whenever $Q(v) = 0$. So, half-linear equations satisfy the homogeneity property of linear equations but not the additivity. Therefore, it is natural to expect that positive solutions of such equations would share some fundamental properties of positive solutions of linear elliptic equations [16, and references therein]. It turns out that this is indeed the case for certain half-linear equations. In fact, the theory of positive solutions of half-linear elliptic equations associated with the p -Laplacian operator Δ_p and a potential

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term V has been studied extensively in recent years (see for example, [2, 3, 9, 17, 18, 19, 20] and the references therein). In particular, the criticality theory and especially the well known Agmon-Allegretto-Piepenbrink (AAP) theorem has been extended from the linear case to such half-linear equations (see [4, Theorem 2.12] and [18, Theorem 2.3]).

In the present work we extend some positivity results of the above mentioned papers concerning half-linear equations with the p -Laplace operator, to the case of half-linear equations with the so called (p, A) -Laplace operator, where A is a given matrix. More precisely, the above mentioned papers study positivity properties of the functional

$$\mathcal{Q}_V(\varphi) := \int_{\Omega} (|\nabla \varphi|^p + V(x)|\varphi|^p) dx \quad \varphi \in C_0^\infty(\Omega),$$

and its associated Euler-Lagrange equation

$$Q_V(u) := \frac{1}{p} \mathcal{Q}'_V(u) = -\Delta_p(u) + V(x)|u|^{p-2}u = 0 \quad \text{in } \Omega, \quad (1.1)$$

where $1 < p < \infty$, Ω is a domain in \mathbb{R}^n , $n \geq 2$. $\Delta_p(u) := \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is the celebrated p -Laplacian, and $V \in L_{\text{loc}}^\infty(\Omega)$ is a real potential.

In our work we study the functional

$$\mathcal{Q}_{A,V}(\varphi) := \int_{\Omega} (|\nabla \varphi|_A^p + V(x)|\varphi|^p) dx \quad \varphi \in C_0^\infty(\Omega),$$

and its Euler-Lagrange equation

$$Q_{A,V}(u) := \frac{1}{p} \mathcal{Q}'_{A,V}(u) = -\nabla \cdot (|\nabla u|_A^{p-2} A(x) \nabla u) + V(x)|u|^{p-2}u = 0 \quad \text{in } \Omega, \quad (1.2)$$

where p , Ω , and V are as assumed above, $A = (a_{ij}) \in L_{\text{loc}}^\infty(\Omega, \mathbb{R}^{n^2})$ is a symmetric and locally uniformly positive definite matrix in Ω , and

$$|\xi|_A^2 := \langle A(x)\xi, \xi \rangle = \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \quad x \in \Omega, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

The aim of the paper is to establish criticality theory for the operator $Q_{A,V}$. In particular, we prove Liouville-type theorems, study the behavior of positive solutions of equation (1.2) near an isolated singularity and near infinity in Ω , and obtain some perturbations results.

It is worth noting that in [10] the authors study, under stronger assumptions on the matrix A , the case where the potential term is missing (i.e. $V = 0$). We generalize some results of this monograph to the case where $V \neq 0$ without assuming [10, (3.4) and (3.5)]. On the other hand, throughout the paper we always assume that $V \in L_{\text{loc}}^\infty(\Omega)$. It would be interesting to extend our results to the case where $V \in L_{\text{loc}}^q(\Omega)$ for an appropriate $q \geq 1$.

The outline of the present paper is as follows. Section 2 is devoted to some preliminaries, while in Section 3 we extend to our setting some fundamental tools, like the *Picone identity* [2, 3, 25], the *Anane-Díaz-Saa identity* [6, 1], and the *simplified energy* [17].

Section 4 is mainly devoted to generalizations of results of J. García-Melián and J. Sabina de Lis [9] concerning the relationships between the principal eigenvalue, the weak and strong

maximum principles, and the solvability of the Dirichlet problem. We note that under the assumptions of [9], solutions of (1.1) are $C^{1,\alpha}$ -smooth and satisfy the boundary point lemma. On the other hand, under our assumptions, solutions of (1.2) are only C^α , and therefore, we need to provide completely new proofs to some of the results in [9].

In Section 5 we extend to our case the well-known AAP theorem dealing with the relationships between positivity properties of the functional $\mathcal{Q}_{A,V}$ and the existence of positive (super)solution of the equation $Q_{A,V}(u) = 0$. Section 6 contains the proof of the Main Theorem (Theorem 6.1) which generalizes [19, Theorem 3.3] and concerns characterizations of critical/subcritical operators. Using the Main Theorem, we study in Section 7 criticality properties of the functional $\mathcal{Q}_{A,V}$, generalizing results of [18, 19, 20].

In Section 8 we use the simplified energy to generalize the Liouville-comparison principle proved in [17, Theorem 1.9], while in Section 9 we prove the existence of a positive minimal Green functions $G_{A,V}^\Omega(x, x_0)$ in the subcritical case, and global minimal solutions of the equation $Q_{A,V}(u) = 0$ in Ω in the critical case. The problem concerning the uniqueness of $G_{A,V}^\Omega(x, x_0)$ remains open for the case $A \neq I$.

2. Preliminaries

In this section we fix our setting and notations, and introduce some basic definitions. Throughout the paper $1 < p < \infty$, and $\Omega \subseteq \mathbb{R}^n$ is a domain. We write $S \Subset \Omega$ if Ω is open, \overline{S} is compact and $\overline{S} \subset \Omega$. By an *exhaustion* of Ω we mean a sequence $\{\Omega_N\}$ of smooth, relatively compact domains such that $x_0 \in \Omega_1$, $\Omega_N \Subset \Omega_{N+1}$, and $\bigcup_{N=1}^\infty \Omega_N = \Omega$.

Let $f, g : \Omega \rightarrow [0, \infty)$. We denote $f \asymp g$ if there exists a positive constant C such that $C^{-1}g \leq f \leq Cg$ in Ω . Also, $f \gtrsim g$ if $f \geq 0$ but $f \neq 0$. For $1 < p < \infty$ we denote $p' := p/(p-1)$ the conjugate index of p . $B_r(x)$ is the open ball of radius r centered at x .

Let us present the regularity assumptions for the operator $Q_{A,V}$ which ensure the validity of the weak and strong Harnack inequalities, and the C^α -regularity of solutions. Throughout our paper we assume (unless otherwise stated) that

$$A : \Omega \rightarrow \mathbb{R}^{n^2} \text{ is a symmetric measurable matrix.} \quad (\text{A})$$

$$\forall K \Subset \Omega \exists \theta_K > 0 \text{ s.t. } \theta_K |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \theta_K^{-1} |\xi|^2 \quad \forall x \in K, \xi \in \mathbb{R}^n. \quad (\text{E})$$

$$V \in L_{\text{loc}}^\infty(\Omega). \quad (\text{V})$$

Remark 2.1. Some results of the paper are proved under stronger regularity assumptions. Indeed, for the up to the boundary $C^{1,\alpha}$ -regularity we need to assume that $A \in C^\alpha$, while for the validity of the boundary point lemma (Corollary 4.6) we need to assume that $A \in C^2$.

We now introduce a formal differential operator $\Delta_{p,A}(u) := \nabla \cdot (|\nabla u|_A^{p-2} A(x) \nabla u)$, called the (p, A) -Laplacian. For $u \in W_{\text{loc}}^{1,p}(\Omega)$ we define:

$$\langle -\Delta_{p,A}(u), \varphi \rangle_\Omega := \int_\Omega |\nabla u|_A^{p-2} A(x) \nabla u \cdot \nabla \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega). \quad (2.1)$$

Remark 2.2. Clearly, for $\Omega' \Subset \Omega$ and $u, v \in W^{1,p}(\Omega')$, the integral $\int_{\Omega'} |\nabla u|_A^{p-2} A(x) \nabla u \cdot \nabla v \, dx$ is well defined, and for such functions we shall still denote

$$\langle -\Delta_{p,A}(u), v \rangle_{\Omega'} := \int_{\Omega'} |\nabla u|_A^{p-2} A(x) \nabla u \cdot \nabla v \, dx.$$

Definition 2.3. A function $v \in W_{\text{loc}}^{1,p}(\Omega)$ is a (*weak*) *solution* of equation (1.2) if

$$\langle Q_{A,V}(v), \varphi \rangle_{\Omega} := \int_{\Omega} (|\nabla v|_A^{p-2} A(x) \nabla v \cdot \nabla \varphi + V(x) |v|^{p-2} v \varphi) \, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega). \quad (2.2)$$

We say that a positive function $v \in W_{\text{loc}}^{1,p}(\Omega)$ is a *supersolution* (*resp.*, *subsolution*) of equation (1.2) if for every nonnegative $\varphi \in C_0^\infty(\Omega)$

$$\int_{\Omega} (|\nabla v|_A^{p-2} A(x) \nabla v \cdot \nabla \varphi + V(x) |v|^{p-2} v \varphi) \, dx \geq 0. \quad (\text{resp. } \leq 0). \quad (2.3)$$

By *uniqueness of positive (super)solutions* of (1.2), we always mean uniqueness up to a multiplicative constant.

In the sequel we need the following elementary lemma.

Lemma 2.4. Let $v \in W_{\text{loc}}^{1,p}(\Omega)$ be a subsolution of the equation (1.2), and let $v_+(x) := \max\{0, v(x)\}$. Then v_+ is also a subsolution of (1.2).

Proof. The proof is a slight modification of the proof of [17, Lemma 2.4] obtained by replacing the Euclidean inner product with the inner product $\langle A(x)\xi, \eta \rangle$ induced by the matrix A . \square

Definition 2.5. The functional $\mathcal{Q}_{A,V}$ is *nonnegative* in Ω (notation: $\mathcal{Q}_{A,V} \geq 0$ in Ω) if

$$\mathcal{Q}_{A,V}(\varphi) = \int_{\Omega} (|\nabla \varphi|_A^p + V(x) |\varphi|^p) \, dx \geq 0 \quad \varphi \in C_0^\infty(\Omega). \quad (2.4)$$

Definition 2.6. Assume that $\mathcal{Q}_{A,V} \geq 0$ in Ω . We say that $\mathcal{Q}_{A,V}$ is *subcritical* in Ω if there exists a nonzero nonnegative continuous function W in Ω such that

$$\mathcal{Q}_{A,V}(\varphi) \geq \int_{\Omega} W |\varphi|^p \, dx \quad \forall \varphi \in C_0^\infty(\Omega). \quad (2.5)$$

A nonnegative functional $\mathcal{Q}_{A,V}$ in Ω which is not subcritical in Ω is called *critical* in Ω . A functional $\mathcal{Q}_{A,V}$ is *supercritical* in Ω if $\mathcal{Q}_{A,V}$ is not nonnegative in Ω .

Definition 2.7. A sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ is a *null sequence* with respect to the nonnegative functional $\mathcal{Q}_{A,V}$ in Ω if $\varphi_k \geq 0$ for all $k \in \mathbb{N}$, and there exists an open set $B \Subset \Omega$ such that

$$\lim_{k \rightarrow \infty} \mathcal{Q}_{A,V}(\varphi_k) = \lim_{k \rightarrow \infty} \int_{\Omega} (|\nabla \varphi_k|_A^p + V |\varphi_k|^p) \, dx = 0, \quad \text{and} \quad \int_B |\varphi_k|^p \, dx = 1. \quad (2.6)$$

We say that a positive function $\phi \in W_{\text{loc}}^{1,p}(\Omega)$ is *Agmon's ground state* (or simply a ground state) of the functional $\mathcal{Q}_{A,V}$ in Ω if ϕ is an $L_{\text{loc}}^p(\Omega)$ limit of a null sequence.

Remark 2.8. The requirement that $\{\varphi_k\} \subset C_0^\infty(\Omega)$, can be weakened by assuming only that $\{\varphi_k\} \subset W_0^{1,p}(\Omega)$, and that $\int_B |\varphi_k|^p dx \asymp 1$ (instead of $\int_B |\varphi_k|^p dx = 1$).

Example 2.9 ([17] Example 1.7). Let $A \in C(\mathbb{R}^N, \mathbb{R}^{n^2})$ be a symmetric, bounded, and uniformly positive definite matrix in \mathbb{R}^n . Consider the functional

$$\mathcal{Q}_{A,0}(\varphi) := \int_{\mathbb{R}^n} |\nabla \varphi|_A^p dx \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

If $p \geq n$, then [14, Theorem 2] and Theorem 6.1 imply that $\mathcal{Q}_{A,0}$ is critical in \mathbb{R}^n and that $\phi = \text{const.} > 0$ is its ground state (see Example 8.3 for an extension of this result).

On the other hand, if $p < n$, then the equation $\mathcal{Q}_{I,0}(u) = -\Delta_p(u) = 0$ in \mathbb{R}^n admits two linearly independent positive supersolutions

$$u(x) := \text{const.}, \quad \text{and} \quad v(x) := \left[1 + |x|^{\frac{p}{p-1}}\right]^{\frac{p-n}{p}}.$$

Hence, Theorem 6.1 implies that $-\Delta_p$ is subcritical in \mathbb{R}^n . For further examples see [17].

We conclude the present section with the following well known compactness result.

Harnack convergence principle. Let $\{\Omega_N\}$ be an exhaustion of Ω . Assume that $\{A_N\}_{N=1}^\infty$ is a sequence of symmetric and positive definite matrices satisfying $A_N \in L^\infty(\Omega_N, \mathbb{R}^{n^2})$ such that the sequence $\{A_N\}_{N=1}^\infty$ converges locally uniformly to a matrix A satisfying conditions (A) and (E). Assume also that $V_N \in L^\infty(\Omega_N)$ satisfy $V_N \rightarrow V$ in $L_{\text{loc}}^\infty(\Omega)$. For each $N \geq 1$, let u_N be a positive solution of the equation

$$\mathcal{Q}_{A_N, V_N}(w) =: -\Delta_{A_N, p}(w) + V_N |w|^{p-2} w = 0 \quad \text{in } \Omega_N, \quad (2.7)$$

satisfying $u_N(x_0) = 1$. By Harnack's inequality and elliptic regularity [21, Chapter 7], and a diagonalization argument, there exist $0 < \beta < 1$, and a subsequence $\{u_{N_k}\}$ of $\{u_N\}$ that converges in $C_{\text{loc}}^\beta(\Omega)$ to a positive solution $u \in W_{\text{loc}}^{1,p}(\Omega)$ of the equation

$$\mathcal{Q}_{A, V}(w) = -\Delta_{A, p}(w) + V |w|^{p-2} w = 0 \quad \text{in } \Omega.$$

3. Picone identity

We start with a simple generalization of Picone identity (cf. [2, 3, 25]).

Proposition 3.1. *Let $v > 0$, $u \geq 0$ be differentiable functions in Ω , and let A satisfy assumptions (A) and (E) in Ω . Denote*

$$L_A(u, v)(x) := |\nabla u|_A^p + (p-1) \frac{u^p}{v^p} |\nabla v|_A^p - p \frac{u^{p-1}}{v^{p-1}} \nabla u \cdot A(x) \nabla v |\nabla v|_A^{p-2}, \quad (3.1)$$

and

$$R_A(u, v)(x) := |\nabla u|_A^p - \nabla \left(\frac{u^p}{v^{p-1}} \right) \cdot A(x) \nabla v |\nabla v|_A^{p-2}. \quad (3.2)$$

Then

$$L_A(u, v)(x) = R_A(u, v)(x) \geq 0 \quad \forall x \in \Omega. \quad (3.3)$$

Moreover, $L_A(u, v) = 0$ a.e. in Ω if and only if $u = kv$ in Ω for some constant $k \geq 0$.

Proof. Use the proof of [2, Theorem 1.1] and just replace the Euclidean inner product $\langle \xi, \eta \rangle$ with the inner product $\langle A(x)\xi, \eta \rangle$ induced by the matrix A . \square

Remark 3.2. By a standard approximation argument, it follows that Proposition 3.1 holds true if $v > 0$, $u \geq 0$ are in $W_{\text{loc}}^{1,p}(\Omega)$ and $uv^{-1} \in L_{\text{loc}}^\infty(\Omega)$.

Proposition 3.3. *Let A and V satisfy assumptions (A), (E) and (V). Let $v \in W_{\text{loc}}^{1,p}(\Omega)$, be a positive solution (resp. supersolution) of (1.2). Then for every $u \geq 0$, $u \in W_{\text{loc}}^{1,p}(\Omega)$ with compact support in Ω such that $\frac{u}{v} \in L_{\text{loc}}^\infty(\Omega)$ we have*

$$\mathcal{Q}_{A,V}(u) = \int_{\Omega} L_A(u, v) \, dx \geq 0. \quad \left(\text{resp. } \mathcal{Q}_{A,V}(u) \geq \int_{\Omega} L_A(u, v) \, dx \geq 0 \right). \quad (3.4)$$

Moreover, (3.4) holds true if $\Omega \Subset \mathbb{R}^n$, A is a bounded measurable symmetric matrix which is uniformly positive definite in Ω , $V \in L^\infty(\Omega)$, $v \in W^{1,p}(\Omega)$ is a positive solution (resp. supersolution) of (1.2), and $0 \leq u \in W_0^{1,p}(\Omega)$ such that $\frac{u}{v} \in L^\infty(\Omega)$.

Proof. From (3.3) we have that

$$0 \leq \int_{\Omega} L_A(u, v) \, dx = \int_{\Omega} |\nabla u|_A^p \, dx - \int_{\Omega} \nabla \left(\frac{u^p}{v^{p-1}} \right) \cdot A(x) \nabla v |\nabla v|_A^{p-2} \, dx. \quad (3.5)$$

Using (2.2) (resp. (2.3)) with $\frac{u^p}{v^{p-1}} \in W_0^{1,p}(\Omega)$, we get that

$$\int_{\Omega} |\nabla v|_A^{p-2} A(x) \nabla v \cdot \nabla \left(\frac{u^p}{v^{p-1}} \right) + V |v|^{p-2} v \frac{u^p}{v^{p-1}} \, dx = 0. \quad (\text{resp. } \geq 0),$$

which imply

$$- \int_{\Omega} \nabla \left(\frac{u^p}{v^{p-1}} \right) \cdot A(x) \nabla v |\nabla v|_A^{p-2} \, dx = \int_{\Omega} V |u|^p \, dx. \quad \left(\text{resp. } \leq \int_{\Omega} V |u|^p \, dx \right). \quad (3.6)$$

Combining (3.5) with (3.6), we arrive at the desired conclusion. \square

Let the hypothesis of Proposition 3.3 be satisfied. Denote $w := \frac{u}{v}$. Then by (3.4),

$$\mathcal{Q}_{A,V}(vw) \stackrel{=}{(\geq)} \int_{\Omega} \left[|v \nabla w + w \nabla v|_A^p - w^p |\nabla v|_A^p - p w^{p-1} v |\nabla v|_A^{p-2} \nabla v \cdot A(x) \nabla w \right] \, dx. \quad (3.7)$$

Similarly, for a such nonnegative *subsolution* v of (1.2) we have,

$$\mathcal{Q}_{A,V}(vw) \leq \int_{\Omega \cap \{v > 0\}} \left[|v \nabla w + w \nabla v|_A^p - w^p |\nabla v|_A^p - p w^{p-1} v |\nabla v|_A^{p-2} \nabla v \cdot A(x) \nabla w \right] \, dx. \quad (3.8)$$

In Proposition 3.3, the functionals $\mathcal{Q}_{A,V}$ and $\int L_A \, dx$ are both nonnegative, but in general, both functionals contain indefinite terms. Therefore, we show now, as in [17, Lemma 2.2], that $\mathcal{Q}_{A,V}$ is equivalent to a *simplified energy* containing only nonnegative terms.

Lemma 3.4. *Let the hypothesis of Proposition 3.3 be satisfied, where v is a positive solution of (1.2). Let $w := \frac{u}{v}$. Then*

$$\mathcal{Q}_{A,V}(vw) \asymp \int_{\Omega} v^2 |\nabla w|_A^2 \left(w |\nabla v|_A + v |\nabla w|_A \right)^{p-2} dx. \quad (3.9)$$

If $v \in W_{\text{loc}}^{1,p}(\Omega)$ is a such nonnegative subsolution of (1.2), then

$$\mathcal{Q}_{A,V}(vw) \leq C \int_{\Omega \cap \{v>0\}} v^2 |\nabla w|_A^2 \left(w |\nabla v|_A + v |\nabla w|_A \right)^{p-2} dx. \quad (3.10)$$

Proof. We claim that the following estimate holds true for all $x \in \Omega$

$$|a + b|_A^p - |a|_A^p - p|a|_A^{p-2} a \cdot A(x)b \asymp |b|_A^2 (|a|_A + |b|_A)^{p-2} \quad \forall a, b \in \mathbb{R}^n, \quad (3.11)$$

where the equivalence constant does not depend on $A(x)$. The proof of (3.11) is similar to the proof of [17, Inequality (2.19)]. Set now $a := w|\nabla v|_A$, $b := v|\nabla w|_A$, and we obtain (3.9) and (3.10) by applying (3.11) to (3.7) and (3.8), respectively. \square

The following lemma is a simple generalization of [9, Lemma 4]. It was proved by Díaz and Saa in [6, Lemma 2] and by Anane in [1, Proposition 1], for the case $A = I$ and $V = 0$.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and assume that A is a symmetric matrix of class $L^\infty(\Omega)$ which is positive definite in Ω , and $V \in L^\infty(\Omega)$. Consider the functional*

$$I_A(u, v) := \left\langle Q_{A,V}(u), \frac{u^p - v^p}{u^{p-1}} \right\rangle_{\Omega} - \left\langle Q_{A,V}(v), \frac{u^p - v^p}{v^{p-1}} \right\rangle_{\Omega},$$

defined on

$$\mathcal{D}(I_A) = \left\{ (u, v) \in (W^{1,p}(\Omega))^2 \mid u, v \geq 0 \text{ in } \Omega, \frac{u}{v}, \frac{v}{u} \in L^\infty(\Omega) \right\},$$

(see Remark 2.2). Then $I_A \geq 0$ on $\mathcal{D}(I_A)$, and $I_A(u, v) = 0$ if and only if $u = kv$ for some constant $k > 0$.

Proof. First we note that

$$\left\langle -\Delta_{p,A}(u), \frac{u^p - v^p}{u^{p-1}} \right\rangle_{\Omega} = \int_{\Omega} |\nabla u|_A^p dx - \int_{\Omega} |\nabla u|_A^{p-2} A(x) \nabla u \cdot \nabla \left(\frac{v^p}{u^{p-1}} \right) dx.$$

So, we have

$$\begin{aligned} I_A(u, v) &= \left\langle Q_{A,V}(u), \frac{u^p - v^p}{u^{p-1}} \right\rangle_{\Omega} - \left\langle Q_{A,V}(v), \frac{u^p - v^p}{v^{p-1}} \right\rangle_{\Omega} = \\ &\quad \left\langle -\Delta_{p,A}(u), \frac{u^p - v^p}{u^{p-1}} \right\rangle_{\Omega} - \left\langle -\Delta_{p,A}(v), \frac{u^p - v^p}{v^{p-1}} \right\rangle_{\Omega} = \\ &= \int_{\Omega} \left(|\nabla u|_A^p - |\nabla u|_A^{p-2} A(x) \nabla u \cdot \nabla \left(\frac{v^p}{u^{p-1}} \right) \right) dx + \int_{\Omega} \left(|\nabla v|_A^p - |\nabla v|_A^{p-2} A(x) \nabla v \cdot \nabla \left(\frac{u^p}{v^{p-1}} \right) \right) dx = \\ &\quad \int_{\Omega} R_A(u, v) dx + \int_{\Omega} R_A(v, u) dx, \end{aligned}$$

and by Proposition 3.1 (see Remark 3.2) we obtain the required result. \square

4. Maximum principles, the principal eigenvalue and the comparison principle

Throughout the present section we assume that $\Omega \subset \mathbb{R}^n$ is a *bounded* domain, and that the coefficients of the operator $Q_{A,V}$ are bounded, and $A(x) = (a_{ij}(x))$ is a symmetric matrix which is positive definite in Ω such that for some $0 < \theta \leq \Theta$ we have

$$\theta|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Theta|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n. \quad (4.1)$$

Let $f \in W^{-1,p'}(\Omega)$. A function $v \in W_{\text{loc}}^{1,p}(\Omega)$ is a weak solution of the equation $Q_{A,V}(u) = f$ in Ω if

$$\int_{\Omega} (|\nabla v|_A^{p-2} A(x) \nabla v \cdot \nabla \varphi + V|v|^{p-2} v \varphi) dx = \langle f, \varphi \rangle \quad \forall \varphi \in C_0^\infty(\Omega). \quad (4.2)$$

For $f \geq 0$, a supersolution of the equation $Q_{A,V}(u) = f$ is defined in a similar fashion.

Denote by

$$a^{ij}(A, x, \nabla u) := \frac{\partial}{\partial u_{x_j}} (|\nabla u|_A^{p-2} A_i(x) \cdot \nabla u),$$

where $A_i(x)$ denotes the i th-row of the matrix $A(x) = (a_{ij}(x))$.

Proposition 4.1. *Let A satisfy (A) and (4.1). Then for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$*

$$\min\{1, p-1\} \theta^{\frac{p}{2}} |\nabla u|^{p-2} |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(A, x, \nabla u) \xi_i \xi_j \leq \max\{1, p-1\} \Theta^{\frac{p}{2}} |\nabla u|^{p-2} |\xi|^2. \quad (4.3)$$

Proof. A direct calculation shows that

$$\left(a^{ij}(A, x, \nabla u) \right) = |\nabla u|_A^{p-2} \left(a_{ij}(x) + \frac{p-2}{|\nabla u|_A^2} (A_i(x) \nabla u) \otimes (A_j(x) \nabla u) \right).$$

Consequently, for all $\xi \in \mathbb{R}^n$ we obtain that

$$\sum_{i,j=1}^n a^{ij}(A, x, \nabla u) \xi_i \xi_j = |\nabla u|_A^{p-2} \left[\xi^T A(x) \xi + \frac{p-2}{|\nabla u|_A^2} \langle A(x) \nabla u, \xi \rangle^2 \right]. \quad (4.4)$$

Obviously, for $p \geq 2$ we have that

$$\sum_{i,j=1}^n a^{ij}(A, x, \nabla u) \xi_i \xi_j \geq \theta |\nabla u|_A^{p-2} |\xi|^2.$$

For $1 < p < 2$, (4.4) and the Cauchy-Schwarz inequality imply

$$\sum_{i,j=1}^n a^{ij}(A, x, \nabla u) \xi_i \xi_j \geq |\nabla u|_A^{p-2} \left[|\xi|_A^2 + \frac{p-2}{|\nabla u|_A^2} |\nabla u|_A^2 |\xi|_A^2 \right] = (p-1) \theta |\nabla u|_A^{p-2} |\xi|^2.$$

For the upper bound, we see that (4.4) readily implies for $1 < p \leq 2$ that

$$\sum_{i,j=1}^n a^{ij}(A, x, \nabla u) \xi_i \xi_j \leq \Theta |\nabla u|_A^{p-2} |\xi|^2.$$

On the other hand, for $p > 2$ we apply the Cauchy-Schwarz inequality to (4.4), and obtain

$$\sum_{i,j=1}^n a^{ij}(A, x, \nabla u) \xi_i \xi_j \leq |\nabla u|_A^{p-2} \left[|\xi|_A^2 + \frac{p-2}{|\nabla u|_A^2} |\nabla u|_A^2 |\xi|_A^2 \right] = (p-1) \Theta |\nabla u|_A^{p-2} |\xi|_A^2.$$

□

Proposition 4.1 is crucial for the following regularity result. Indeed, since $A(x)$ satisfies (4.3), Lemma 4.2 below follows from [21, Theorem 6.1.1], [11, Theorem 7.2, p. 290], and [13, Theorem 1]. We have

Lemma 4.2. *Let Ω be a bounded domain in \mathbb{R}^n of class $C^{2,\alpha}$. Assume that the matrix A is a bounded measurable symmetric matrix which is uniformly positive definite in Ω , and $V \in L^\infty(\Omega)$. Let $u \in W^{1,p}(\Omega)$ be a weak solution of the Dirichlet problem*

$$\begin{cases} Q_{A,V}(w) = f \geq 0 & \text{in } \Omega, \\ w = f_1 \geq 0 & \text{on } \partial\Omega, \end{cases} \quad (4.5)$$

where $f \in L^\infty(\Omega)$, and $f_1 \in C^\alpha(\partial\Omega)$ with $0 < \alpha \leq 1$. Then $u \in L^\infty(\Omega) \cap C^\alpha(\bar{\Omega})$.

If in addition the matrix $A \in C^\beta(\bar{\Omega})$ and $f_1 \in C^{1,\beta}(\partial\Omega)$ for some $0 < \beta \leq 1$, then there exists $0 < \gamma \leq 1$ such that $u \in C^{1,\gamma}(\bar{\Omega})$.

Next, we generalize the results of J. García-Melián, and J. Sabina de Lis [9, Section 2] concerning the relationships between the principal eigenvalue, the weak and strong maximum principles, and the solvability of the Dirichlet problem.

Definition 4.3. By the *strong maximum principle* (SMP) of a quasilinear equation $M(u) = 0$ in Ω we mean the following statement: if u is a nonnegative supersolution of the equation $M(u) = 0$ in Ω with $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u = 0$ in Ω .

Since the SMP is a local property, the weak Harnack inequality (see [21, theorems 7.1.2 and 7.4.1]) clearly implies:

Lemma 4.4 (SMP). *Assume that the matrix A and the potential V satisfy conditions (A), (E), and (V). If $0 \leq u \in W_{\text{loc}}^{1,p}(\Omega) \cap C(\Omega)$ satisfies the differential inequality*

$$-\Delta_{p,A}(u) + V|u|^{p-2}u \geq 0 \quad \text{in } \Omega, \quad (4.6)$$

and $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u = 0$ in Ω .

We turn now to the boundary point lemma for our equation. First, we prove a boundary point lemma that holds for nonnegative functions satisfying the differential inequality

$$-\Delta_{p,A}(u) + f(u) \geq 0 \quad \text{in } \Omega, \quad (4.7)$$

where, $f = f(u)$ satisfies the following condition (F):

$$f \in C(0, \infty), \quad f(0) = 0, \quad \text{and } f \text{ is nondecreasing on the interval } (0, \delta), \quad \text{where } \delta > 0. \quad (\text{F})$$

Theorem 4.5 (Boundary point lemma). *Suppose that A satisfies assumptions (A) and (E), and f satisfy condition (F). Assume that Ω is of class C^2 . Suppose that $f(s) > 0$ for $s \in (0, \delta)$. Let $x_1 \in \partial\Omega$ satisfy the interior sphere condition, and assume that the matrix A is of class C^2 in a closed relative neighborhood of x_1 . Let $u \in C^1(\bar{\Omega})$ be a positive solution of (4.7) such that $u(x_1) = 0$. Then $\frac{\partial u(x_1)}{\partial \nu} < 0$, where ν is the outer normal to $\partial\Omega$.*

Proof. Let $\mathcal{A}(s) = s^{p-2}$, and introduce the Riemannian metric $g^{ij}(x) := a_{ij}(x)$ induced by the matrix A , and let $\mathbf{s}(x) := \text{dist}(x, x_0)$ be the induced geodesic distance from $x_0 \in \Omega$. For certain “radial” functions v we need to estimate the expression

$$\Delta_{p,A}(v) - f(v) = \partial_{x^i} \{ g^{ij}(x) \mathcal{A}(|\nabla v|_g) \partial_{x^j} v \} - f(v)$$

in the annular domain $G_S := \{x \in \Omega \mid S/2 < \mathbf{s}(x) < S\}$ centered at x_0 . As in [21, p. 220], set $t := S - \mathbf{s}(x)$, $\Phi(t) := t\mathcal{A}(t)$, and let w be the (unique) solution of the problem

$$\begin{cases} -\frac{1}{(S-t)^k} [(S-t)^k \Phi(w'(t))] + f(w(t)) = 0 & \text{in } (0, S/2), \\ w(0) = 0, w(S/2) = m > 0, \end{cases}$$

where $k \in \mathbb{N}$ will be determined later. The existence and uniqueness of w is guaranteed by [21, Lemma 4.2.1]. Moreover, by [21, Lemma 4.2.2], $w > 0$ in $(0, S/2]$, and $w' > 0$ in $[0, S/2]$.

Consider the “radial” function $v(x) := w(t)$. By restricting the boundary value $v = m$ at $\partial B_{S/2}(x_0)$ to be sufficiently small, one can maintain $\sup |\nabla v|_g \leq \Theta |\nabla v| \leq 1$.

Using the summation convention, we have for the “radial” function v in G_S :

$$\begin{aligned} \Delta_{p,A}(v) - f(v) &= \partial_{x^i} \{ g^{ij}(x) \mathcal{A}(|\nabla v|_g) \partial_{x^j} v \} - f(v) = \\ &= g^{ij}(x) \partial_{x^j} \mathbf{s}(x) \partial_{x^i} \mathbf{s}(x) [\Phi(w')] - \partial_{x^i} \{ g^{ij}(x) \partial_{x^j} \mathbf{s}(x) \} \Phi(w') - f(w) = \\ &= [\Phi(w')] - \partial_{x^i} \{ g^{ij}(x) \partial_{x^j} \mathbf{s}(x) \} \Phi(w') - f(w). \end{aligned}$$

For S small, we have $\mathbf{s} \in C^2(G_S)$, and by [22, Corollary 1.2], there exists $\bar{k} \in \mathbb{N}$ such that

$$\Delta \mathbf{s} = \frac{1}{\sqrt{g(x)}} \partial_{x^i} \left\{ \sqrt{g(x)} g^{ij}(x) \partial_{x^j} \mathbf{s}(x) \right\} \leq \frac{\bar{k}}{\mathbf{s}(x)}. \quad (4.8)$$

A direct calculation shows that

$$\begin{aligned} \partial_{x^i} \{ g^{ij}(x) \partial_{x^j} \mathbf{s}(x) \} &= \frac{1}{\sqrt{g(x)}} \partial_{x^i} \left\{ \sqrt{g(x)} g^{ij}(x) \partial_{x^j} \mathbf{s}(x) \right\} - \frac{1}{\sqrt{g(x)}} \{ g^{ij}(x) \partial_{x^j} \mathbf{s}(x) \} \partial_{x^i} \left\{ \sqrt{g(x)} \right\} = \\ &= \Delta \mathbf{s} - \frac{1}{\sqrt{g(x)}} \{ g^{ij}(x) \partial_{x^j} \mathbf{s}(x) \} \partial_{x^i} \left\{ \sqrt{g(x)} \right\}. \end{aligned}$$

Since $|\nabla \mathbf{s}(x)|$ is bounded, $g^{ij} \in C^2(\Omega)$, and G_S is compact, we obtain using (4.8) that there exists constant $k \in \mathbb{N}$ such that

$$\Delta_{p,A}(v(x)) - f(v(x)) \geq [\Phi(w')] - \frac{k}{\mathbf{s}} \Phi(w') - f(w) = 0 \quad x \in G_S. \quad (4.9)$$

We claim that $\Delta_{p,A}$ satisfies the monotonicity condition needed for the of the comparison principle [21, Theorem 3.4.1]. Indeed, $\hat{\mathcal{A}}(x, \xi) := g^{ij}(x)\mathcal{A}(|\xi|_g)\xi$ satisfies [21, (2.4.3)]:

$$\begin{aligned} \langle \hat{\mathcal{A}}(x, \xi) - \hat{\mathcal{A}}(x, \eta), \xi - \eta \rangle = \\ \mathcal{A}(|\xi|_g)|\xi|_g^2 - \mathcal{A}(|\xi|_g) \langle [g^{ij}]\xi, \eta \rangle - \mathcal{A}(|\eta|_g) \langle [g^{ij}]\eta, \xi \rangle + \mathcal{A}(|\eta|_g)|\eta|_g^2 \geq \\ (\mathcal{A}(|\xi|_g)|\xi|_g - \mathcal{A}(|\eta|_g)|\eta|_g) (|\xi|_g - |\eta|_g) = (|\xi|_g^{p-1} - |\eta|_g^{p-1}) (|\xi|_g - |\eta|_g) \geq 0, \end{aligned} \quad (4.10)$$

where we used the Cauchy-Schwarz inequality $|\langle [g^{ij}]\eta, \xi \rangle| \leq |\eta|_g|\xi|_g$ and the monotonicity of $\Phi(t) = t^{p-1}$. Moreover, equality holds in (4.10) if and only if $\xi = \eta$.

Choosing m sufficiently small, it follows from the comparison principle [21, Theorem 3.4.1] that $u \geq v$ in G_S . Therefore, $\frac{\partial u}{\partial \nu} \leq \frac{\partial v}{\partial \nu} = -w'(t) < 0$. □

Let $M \geq \sup_{x \in \Omega} \{|V(x)|\}$. Then $f(u) := Mu^{p-1}$ satisfies condition (F), and a nonnegative supersolution u of the equation $Q_{A,V}(w) = 0$ in Ω clearly satisfies

$$-\Delta_{p,A}(u) + Mu^{p-1} \geq 0.$$

Hence, Theorem 4.5 implies the following boundary point lemma for the operator $Q_{A,V}$:

Corollary 4.6 (Boundary point lemma). *Suppose that the matrix A and the potential V satisfy assumptions (A), (E), and (V). Then the following boundary point lemma holds true (and also the SMP): Let $u \not\geq 0$ satisfy the differential inequality*

$$-\Delta_{p,A}(u) + V|u|^{p-2}u \geq 0 \quad \text{in } \Omega. \quad (4.11)$$

Then $u > 0$ in Ω . Suppose further that $u(x_1) = 0$, where $x_1 \in \partial\Omega$ satisfies the interior sphere condition, and in a closed relative neighborhood Ω' of x_1 : $u \in C^1$, V is bounded, and the matrix A is uniformly positive definite and of class C^2 . Then $\frac{\partial u(x_1)}{\partial \nu} < 0$, where ν is the outer normal to $\partial\Omega$.

Next, we define a principal eigenvalue of the corresponding Dirichlet eigenvalue problem.

Definition 4.7. For $\Omega \subset \mathbb{R}^n$, consider the eigenvalue problem

$$\begin{cases} Q_{A,V}(u) = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), u \neq 0. \end{cases} \quad (4.12)$$

We say that $\lambda \in \mathbb{R}$ is a *principal eigenvalue* of the operator $Q_{A,V}$ in Ω if there exists a nonnegative function u satisfying (4.12) (such u is called a *principal eigenfunction*).

Remark 4.8. Let $A \in C^2(\bar{\Omega}, \mathbb{R}^{n^2})$ be a symmetric positive definite matrix, and let $V \in L^\infty(\Omega)$, where Ω is a smooth bounded domain. If ψ and ϕ are *nonnegative* eigenfunctions of problem (4.12), then by the boundary point lemma (Corollary 4.6) we have $\frac{\psi}{\phi} \in L^\infty(\Omega)$.

It turns out that if $\Omega \Subset \mathbb{R}^n$, then $Q_{A,V}$ admits a principal eigenvalue λ_1 defined by (4.13) with a principal eigenfunction which is a minimizer of the variational problem:

$$\lambda_1 := \lambda_1(Q_{A,V}, \Omega) = \lambda_1(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega (|\nabla u|_A^p + V(x)|u|^p) dx}{\int_\Omega |u|^p dx}. \quad (4.13)$$

Lemma 4.9. *Assume that the matrix $A \in L^\infty(\Omega, \mathbb{R}^{n^2})$ is a symmetric uniformly positive definite matrix in a bounded domain Ω , and $V \in L^\infty(\Omega)$. Then the eigenvalue problem (4.12) admits a principal eigenvalue λ with a principal eigenfunction $\phi \in W_0^{1,p}(\Omega)$. Such a principal eigenpair is given by λ_1 , and a minimizer of (4.13). Furthermore, all eigenfunctions with eigenvalue λ_1 does not vanish in Ω .*

Moreover, if $A \in C^2(\bar{\Omega})$ and $\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$, then $\phi \in C^{1,\beta}(\bar{\Omega})$ for some $0 < \beta < 1$, and $\frac{\partial \phi}{\partial \nu} < 0$ on $\partial\Omega$.

Proof. We repeat, with necessary changes, the proof of [9, Lemma 3]. Since $V \in L^\infty(\Omega)$, we may assume that $V(x) \geq 0$. Otherwise, replace V with $V_M(x) := V(x) + M \geq 0$ and $\lambda_M := \lambda + M$.

Now, if $V \geq 0$, then the functional $\mathcal{Q}_{A,V}$ is sequentially weakly lower semicontinuous in $W_0^{1,p}(\Omega)$, and coercive in $\mathcal{V} := \{u \in W_0^{1,p}(\Omega) \mid \int_\Omega |u|^p dx = 1\}$. Hence, the infimum λ_1 in (4.13) is attained. In particular, there exists $\phi \in \mathcal{V}$ such that ϕ is a weak solution of the equation

$$-\Delta_{p,A}(u) + V(x)|u|^{p-2}u = \lambda_1|u|^{p-2}u. \quad (4.14)$$

Since $|\nabla(|\phi|)| \leq |\nabla\phi|$, we get that $|\phi|$ is also a minimizer of (4.13) and hence it is a nonnegative weak solution of equation (4.14). Thus, by the Harnack inequality either $|\phi| > 0$ or $|\phi| = 0$, and consequently, ϕ does not vanish in Ω , and has a definite sign in Ω . Furthermore, by the same reasoning all eigenfunctions with eigenvalue λ_1 does not vanish in Ω .

Moreover, if $A \in C^\alpha(\bar{\Omega})$, then by Lemma 4.2, $\phi \in C^{1,\beta}(\bar{\Omega})$. Finally, if $A \in C^2(\bar{\Omega})$, then by the SMP and the boundary point lemma (Corollary 4.6), we have that $\frac{\partial |\phi|}{\partial \nu} < 0$ on $\partial\Omega$. \square

Using our previous results we extend the main theorem of J. García-Melián, and J. Sabina de Lis [9, Theorem 2]. We have:

Theorem 4.10. *Let A be a bounded symmetric matrix which is uniformly positive definite in a $C^{1,\alpha}$ -bounded domain Ω ($0 < \alpha \leq 1$), and $V \in L^\infty(\Omega)$. Consider the following assertions.*

- (i) $\mathcal{Q}_{A,V}(u)$ satisfies the following weak maximum principle (WMP): *If $u \in W^{1,p}(\Omega)$ is a solution of the equation $\mathcal{Q}_{A,V}(u) = f \geq 0$ in Ω with some $f \in L^\infty(\Omega)$, and satisfies $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω .*
- (ii) $\mathcal{Q}_{A,V}(u)$ satisfies the following version of the strong maximum principle: *If $u \in W^{1,p}(\Omega)$ is a solution of the equation $\mathcal{Q}_{A,V}(u) = f \geq 0$ in Ω with some $f \in L^\infty(\Omega)$, and satisfies $u \geq 0$ on $\partial\Omega$, and $u \neq 0$ in Ω , then $u > 0$ in Ω .*
- (iii) $\lambda_1(\Omega) > 0$, where λ_1 is defined by (4.13).
- (iv) *There exists a positive strict supersolution $v \in W_0^{1,p}(\Omega)$ of $\mathcal{Q}_{A,V}(w) = 0$ in Ω , with $\mathcal{Q}_{A,V}(v) \in L^\infty(\Omega)$, i.e., $\mathcal{Q}_{A,V}(v) = f$ in Ω , where $f \in L^\infty(\Omega)$, $f \not\geq 0$.*
- (iv') *There exists a positive strict supersolution $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ of $\mathcal{Q}_{A,V}(w) = 0$ in Ω , with $\mathcal{Q}_{A,V}(v) \in L^\infty(\Omega)$, i.e., $\mathcal{Q}_{A,V}(v) = f$ in Ω , where $f \in L^\infty(\Omega)$, $f \not\geq 0$.*
- (v) *For each nonnegative $f \in L^\infty(\Omega)$ there exists a nonnegative weak solution $u \in W_0^{1,p}(\Omega)$ of the problem $\mathcal{Q}_{A,V}(w) = f$ in Ω , and $w = 0$ on $\partial\Omega$.*

Then (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (iv') , and (iii) \Rightarrow (v) \Rightarrow (iv').

Proof. (i) \Rightarrow (ii). Let $u \in W^{1,p}(\Omega)$ be a solution of the equation $Q_{A,V}(w) = f \geq 0$ in Ω with $f \in L^\infty(\Omega)$ such that $u \geq 0$ on $\partial\Omega$, and $u \neq 0$ in Ω . The WMP (i) implies that $u \geq 0$ in Ω . Hence, the strong maximum principle of Lemma 4.4 implies that $u > 0$ in Ω .

(ii) \Rightarrow (iii). Assume that $\lambda_1 \leq 0$, and let $\phi > 0$ be the corresponding principal eigenfunction. Then $\psi := -\phi$ satisfies

$$-\Delta_{p,A}(\psi) + V|\psi|^{p-2}\psi = \lambda_1|\psi|^{p-2}\psi \geq 0,$$

and $\psi = -\phi = 0$ on $\partial\Omega$. By (ii) we have $\psi > 0$ in Ω , hence, $\phi < 0$, a contradiction.

(iii) \Rightarrow (i). Assume, that $u \in W^{1,p}(\Omega)$ is a solution of the equation $Q_{A,V}(w) = f \geq 0$ in Ω and $u \geq 0$ on $\partial\Omega$, with $f \in W^{-1,p'}(\Omega)$, and $\langle f, v \rangle \geq 0$ for each nonnegative $v \in W_0^{1,p}(\Omega)$. Denote $u_-(x) := \min\{u(x), 0\}$, hence $u_- \in W_0^{1,p}(\Omega)$. Consequently,

$$\begin{aligned} \int_{\Omega} (|\nabla u_-|_A^p + V|u_-|^p) dx &= \int_{\Omega} (|\nabla u_-|_A^{p-2} A(x) \nabla u_- \cdot \nabla u_- + V|u_-|^{p-2} u_- u_-) dx = \\ &= \int_{\Omega} (|\nabla u|_A^{p-2} A(x) \nabla u \cdot \nabla u_- + V|u|^{p-2} u u_-) dx = \langle f, u_- \rangle \leq 0 < \lambda_1. \end{aligned}$$

In light of the definition of λ_1 , we obtain that $u_- = 0$, so, $u \geq 0$ in Ω .

(iii) \Rightarrow (iv). The principal eigenfunction is a desired positive supersolution.

(iv) \Rightarrow (iv'). This implication is trivial.

(iii) \Rightarrow (v). Let $f \in W^{-1,p'}(\Omega)$, and define a functional

$$\mathcal{J}_f(u) = \int_{\Omega} (|\nabla u|_A^p + V|u|^p - p f u) dx.$$

Since $\lambda_1 > 0$, the functional $\mathcal{J}_f(u)$ is coercive in $W_0^{1,p}(\Omega)$. Hence, for each $0 \leq f \in L^\infty(\Omega)$ there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of problem (4.5) (with $f_1 = 0$). Moreover, since (iii) \Rightarrow (i) and (ii), it follows that $u \geq 0$ in Ω , and if $f \neq 0$ it follows that $u > 0$ in Ω .

(v) \Rightarrow (iv'). Follows directly from the SMP. \square

Remark 4.11. Later (in Corollary 6.6) we show that (iv') \Rightarrow (iii). Hence, all the assertions of Theorem 4.10 are in fact equivalent.

For the Dirichlet problem (4.5) we have the following uniqueness result.

Theorem 4.12. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2,\alpha}$ -domain, $0 < \alpha < 1$, $A \in C^2(\bar{\Omega}, \mathbb{R}^{n^2})$ is positive definite, and $V \in L^\infty(\Omega)$. Then for nonnegative $f \in L^\infty(\Omega)$ and $f_1 \in C^{1,\alpha}(\partial\Omega)$ there exists at most one nonzero nonnegative weak solution $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ of problem (4.5) (up to a multiplicative constant if $f = f_1 = 0$).

Moreover, if $f_1 > 0$, $f_1 \in C^\alpha(\partial\Omega)$, then the uniqueness of (4.5) holds even if A is only a bounded measurable symmetric matrix which is uniformly positive definite.

Proof. Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\bar{\Omega})$ be nonnegative solutions of (4.5). In light of [11, Theorem 7.2, p. 290], $u, v \in C^\alpha(\bar{\Omega})$. By the SMP (Lemma 4.4), any such solution is positive

in Ω . Hence, (using the boundary point lemma, in case $f_1 \not\geq 0$ and $A \in C^2$), we have $\frac{v}{u}, \frac{u}{v} \in L^\infty(\Omega)$. Therefore, $(u, v) \in \mathcal{D}(I)$, and by Lemma 3.5 we have

$$\begin{aligned} 0 &\leq \left\langle -\Delta_{p,A}(u), \frac{u^p - v^p}{u^{p-1}} \right\rangle_\Omega - \left\langle -\Delta_{p,A}(v), \frac{u^p - v^p}{v^{p-1}} \right\rangle_\Omega = \\ &\left\langle f - V|u|^{p-2}u, \frac{u^p - v^p}{u^{p-1}} \right\rangle_\Omega - \left\langle f - V|v|^{p-2}v, \frac{u^p - v^p}{v^{p-1}} \right\rangle_\Omega = \int_\Omega f \frac{(u^p - v^p)(v^{p-1} - u^{p-1})}{u^{p-1}v^{p-1}} dx \leq 0. \end{aligned}$$

It follows that $I(u, v) = 0$, and Lemma 3.5 implies that $u = kv$ for some positive k . If either $f \not\geq 0$ or $f_1 \not\geq 0$, it follows that $k = 1$, and hence $u = v$. \square

The following is an extension of the weak comparison principle (WCP) [9, Theorem 5].

Theorem 4.13. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2,\alpha}$ -domain, $0 < \alpha < 1$, A is a symmetric bounded matrix which is uniformly positive definite in Ω , and $V \in L^\infty(\Omega)$. Assume that $\lambda_1 > 0$, where λ_1 is defined by (4.13). Let $u_i \in W^{1,p}(\Omega) \cap C^\alpha(\bar{\Omega})$ satisfy $Q_{A,V}(u_i) \in L^\infty(\Omega)$, where $i = 1, 2$. Suppose further that the following inequalities are satisfied*

$$\begin{cases} Q_{A,V}(u_1) \leq Q_{A,V}(u_2) & \text{in } \Omega, \\ Q_{A,V}(u_2) \geq 0 & \text{in } \Omega, \\ u_1 \leq u_2 & \text{on } \partial\Omega, \\ u_2 > 0 & \text{on } \partial\Omega. \end{cases}$$

Then, $u_1 \leq u_2$ in Ω . Moreover, if the conditions for the validity of the boundary point lemma are satisfied, then the conclusion holds true even if $u_2 \geq 0$ on $\partial\Omega$.

Proof. Since $u_2 > 0$ in $\bar{\Omega}$, there exists a constant $c > 1$ such that $u_1 < cu_2$ in Ω . Set $g := Q_{A,V}(u_2)$, $g_2 = u_2|_{\partial\Omega}$, and consider the Dirichlet problem

$$\begin{cases} Q_{A,V}(w) = g & \text{in } \Omega, \\ w = g_2 & \text{on } \partial\Omega. \end{cases} \quad (4.15)$$

Clearly, u_1 is a subsolution and cu_2 is a supersolution of (4.15). Therefore, by the sub/supersolution technique (see [5, Theorem 4.14, p. 272]), there exists a weak solution $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (4.15), satisfying $u_1 \leq v \leq cu_2$, and $v = g_2|_{\partial\Omega}$ in the sense of traces. Moreover, Theorem 4.10 implies that $v > 0$ in Ω . In light of Lemma 4.2, $v \in C^\alpha(\bar{\Omega})$ (or even $v \in C^{1,\alpha}(\bar{\Omega})$ if $A \in C^\alpha(\bar{\Omega})$). Consequently, by uniqueness (Theorem 4.12), $v = u_2$. Thus, $u_1 \leq u_2$. \square

Problem 4.14. Prove the WCP assuming only $v_2 \geq 0$ on $\partial\Omega$ (without using the boundary point lemma).

5. The Agmon-Allegretto-Piepenbrink (AAP) theory

In the present section we generalize the AAP theorem claiming that $Q_{A,V} \geq 0$ in Ω if and only if the equation $Q_{A,V}(u) = 0$ admits a positive (super)solution in Ω . First, we need to prove the strict monotonicity of λ_1 with respect to the domain (see [2, Theorem 2.3]).

Lemma 5.1. *Let $\Omega_1 \Subset \Omega_2 \Subset \Omega$ be smooth bounded domains, and suppose that A and V satisfy assumptions (A), (E), and (V) in Ω , and $\mathcal{Q}_{A,V} \geq 0$ in Ω_2 . Then $\lambda_1(\Omega_1) > \lambda_1(\Omega_2) \geq 0$.*

Proof. It follows from (4.13) that $\lambda_1(\Omega_2) \geq 0$. Let $\phi_i \in W_0^{1,p}(\Omega_i)$ be normalized principal eigenfunctions of the operator $Q_{A,V}$ in Ω_i with eigenvalues $\lambda_1(\Omega_i)$, $i = 1, 2$. Let $\{\varphi_k\} \subset C_0^\infty(\Omega_1)$ be a nonnegative minimizing sequence that converges to ϕ_1 in $W_0^{1,p}(\Omega_1)$. By Picone identity we have

$$\begin{aligned} 0 &\leq \int_{\Omega_2} L_A(\varphi_k, \phi_2) dx = \int_{\Omega_1} L_A(\varphi_k, \phi_2) dx = \int_{\Omega_1} R_A(\varphi_k, \phi_2) dx = \\ &\int_{\Omega_1} |\nabla \varphi_k|_A^p dx - \int_{\Omega_1} \nabla \left(\frac{\varphi_k^p}{\phi_2^{p-1}} \right) \cdot A(x) \nabla \phi_2 |\nabla \phi_2|_A^{p-2} dx = \int_{\Omega_1} |\nabla \varphi_k|_A^p dx + \int_{\Omega_1} V(x) \varphi_k^p dx - \lambda_1(\Omega_2) \int_{\Omega_1} \varphi_k^p dx. \end{aligned}$$

Letting $k \rightarrow \infty$, and using the Fatou's lemma, we arrive at

$$0 \leq \int_{\Omega_1} L_A(\phi_1, \phi_2) dx \leq \lim_{k \rightarrow \infty} \int_{\Omega_1} L_A(\varphi_k, \phi_2) dx = \left(\lambda_1(\Omega_1) - \lambda_1(\Omega_2) \right) \int_{\Omega_1} \phi_1^p dx.$$

So, $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$. If $\lambda_1(\Omega_1) = \lambda_1(\Omega_2)$, then $L_A(\phi_1, \phi_2) = 0$ a. e. in Ω_1 . Hence, Proposition 3.1 implies that there is a constant $c > 0$ such that $\phi_2|_{\Omega_1} = c\phi_1$, and this is impossible since $\phi_2 > 0$ on $\partial\Omega_1$ (by the Harnack inequality). Thus, $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$. \square

Using our earlier results, we extend now the AAP-type theorem.

Theorem 5.2 (AAP-type theorem). *Suppose that A and V satisfy assumptions (A), (E), and (V) in a domain $\Omega \subset \mathbb{R}^n$. Then the following assertions are equivalent:*

- (i) *The functional $\mathcal{Q}_{A,V}$ is nonnegative on $C_0^\infty(\Omega)$.*
- (ii) *The equation $Q_{A,V}(u) = 0$ in Ω admits a positive solution.*
- (iii) *The equation $Q_{A,V}(u) = 0$ in Ω admits a positive supersolution.*

Proof. (i) \implies (ii). Assume that $\mathcal{Q}_{A,V} \geq 0$ on $C_0^\infty(\Omega)$, and let $\{\Omega_N\}$ be an exhaustion of Ω . By Lemma 5.1, $\lambda_1(\Omega_N) > 0$ for all $N \in \mathbb{N}$. Let $f_N \in C_0^\infty(\Omega_N \setminus \Omega_{N-1})$ be a nonnegative nonzero function. Theorem 4.10 implies the existence of a positive solution v_N of the problem

$$\begin{cases} Q_{A,V}(w) = f_N & \text{in } \Omega_N, \\ w = 0 & \text{on } \Omega_N. \end{cases}$$

Set $u_N(x) := \frac{v_N(x)}{v_N(x_0)}$. By the Harnack convergence principle, $\{u_N\}$ admits a subsequence which converges locally uniformly to a positive solution u of the equation $Q_{A,V}(w) = 0$ in Ω .

(ii) \implies (iii). This implication is trivial.

(iii) \implies (i). Suppose that v is a positive supersolution of (1.2) in Ω . Then Proposition 3.3 implies that $\mathcal{Q}_{A,V}(\varphi) \geq 0$ for all nonnegative $\varphi \in C_0^\infty(\Omega)$. Since $\mathcal{Q}_{A,V}(u) = \mathcal{Q}_{A,V}(|u|)$ on $C_0^\infty(\Omega)$, a standard approximation argument shows that $\mathcal{Q}_{A,V} \geq 0$ on $C_0^\infty(\Omega)$. \square

6. The Main Theorem

This section is devoted to the following result which generalizes [19, Theorem 3.3].

Theorem 6.1 (Main Theorem). *Suppose that the matrix A and the potential V satisfy conditions (A), (E) and (V). If $p < 2$ assume further that $A \in C^\alpha(\Omega)$. Suppose that $\mathcal{Q}_{A,V}$ is nonnegative on $C_0^\infty(\Omega)$. Then*

1. *Any ground state ϕ is a positive solution of (1.2).*
2. *$\mathcal{Q}_{A,V}$ is critical in Ω if and only if $\mathcal{Q}_{A,V}$ admits a null sequence. Moreover, there exists a null sequence that converges locally uniformly in Ω to the ground state.*
3. *$\mathcal{Q}_{A,V}$ admits a null sequence if and only if (1.2) admits a unique positive continuous supersolution.*
4. *$\mathcal{Q}_{A,V}$ is subcritical in Ω if and only if there exists a strictly positive continuous function W such that $\mathcal{Q}_{A,V-W}$ is nonnegative on $C_0^\infty(\Omega)$.*
5. *If $\mathcal{Q}_{A,V}$ admits a ground state ϕ , then the following Poincaré type inequality holds: There exists $0 < W \in C(\Omega)$ such that for every $\psi \in C_0^\infty(\Omega)$ satisfying $\int_\Omega \psi \phi \, dx \neq 0$ there exists a constant $C > 0$ so that the following inequality holds:*

$$\mathcal{Q}_{A,V}(\varphi) + C \left| \int_\Omega \varphi \psi \, dx \right|^p \geq C^{-1} \int_\Omega W(x) |\varphi|^p \, dx \quad \forall \varphi \in C_0^\infty(\Omega). \quad (6.1)$$

The proof of Theorem 6.1 is based on the proofs in [18, Section 3] and [20, Theorem 4.3]. First we need to generalize Lemma 3.1 and Lemma 3.2 in [18].

Let $B \subset \Omega$ be a nonempty open set, and set

$$c_B := \inf_{\substack{\varphi \in C_0^\infty(\Omega) \\ \int_B |\varphi|^p \, dx = 1}} \mathcal{Q}_{A,V}(\varphi) = \inf_{\substack{0 \leq \varphi \in C_0^\infty(\Omega) \\ \int_B |\varphi|^p \, dx = 1}} \mathcal{Q}_{A,V}(\varphi). \quad (6.2)$$

Clearly, the criticality of $\mathcal{Q}_{A,V}$ in Ω implies that $c_B = 0$ for any nonempty open set in Ω .

Lemma 6.2. *If for every open set $B \Subset \Omega$, $c_B > 0$, then there exists a continuous positive function W , such that*

$$\mathcal{Q}_{A,V}(\varphi) \geq \int_\Omega W(x) |\varphi(x)|^p \, dx \quad \forall \varphi \in C_0^\infty(\Omega). \quad (6.3)$$

Proof. The proof is obtained by a partition of unity argument as in [18, Lemma 3.1]. \square

Lemma 6.3. *Suppose that $\mathcal{Q}_{A,V} \geq 0$ in Ω , where A and V satisfy conditions (A), (E) and (V). If $p < 2$ assume further that $A \in C^\alpha(\Omega)$. If there exists a nonempty open set $B \Subset \Omega$ such that $c_B = 0$, then $\mathcal{Q}_{A,V}$ admits a ground state ϕ . Moreover, ϕ is the unique positive supersolution of the equation $\mathcal{Q}_{A,V}(u) = 0$ in Ω .*

Proof. Since $c_B = 0$, it follows that there exists a null sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$, such that $\varphi_k \geq 0$, and $\int_B |\varphi_k|^p \, dx = 1$. Fix a positive (super)solution $v \in W_{\text{loc}}^{1,p}(\Omega)$ of (1.2).

Assume first that $p \geq 2$. So, we may assume that $v \in C^\alpha(\Omega)$. Denote $w_k := \varphi_k/v$. By Lemma 3.4 we have $\int_\Omega v^p |\nabla w_k|_A^p \, dx \leq C \mathcal{Q}_{A,V}(\varphi_k) \rightarrow 0$. Hence, $\nabla(w_k) \rightarrow 0$ in $L_{\text{loc}}^p(\Omega)$.

Poincaré inequality in C^1 -subdomains $\omega \Subset \Omega$, implies that $w_k \rightarrow \text{const}$ in $W_{\text{loc}}^{1,p}(\Omega)$. Consequently, there exists $c \geq 0$ such that (up to a subsequence) $\varphi_k \rightarrow cv$ a.e. in Ω , and also in $L_{\text{loc}}^p(\Omega)$. Thus, $c^{-p} = \int_B v^p dx > 0$. It follows that any null sequence $\{\varphi_k\}$ converges to the same positive supersolution v . Hence, v is the unique positive solution, and also the unique positive supersolution of the equation $Q_{A,V}(u) = 0$ in Ω .

Assume now that $1 < p < 2$. In this case the proof is more involved and consists of three steps (see the proof of [18, Lemma 3.2]).

Step 1. By our assumption, we may assume that $v \in C^{1,\alpha}(\Omega)$. Let $\omega \Subset \Omega$ be an open connected set containing B , and let $\omega' \subset \omega$. By (3.1) we have that

$$\int_{\omega'} L_A(\varphi_k, v) dx = \int_{\omega'} |\nabla \varphi_k|_A^p dx + (p-1) \int_{\omega'} \left(\frac{\varphi_k}{v} \right)^p |\nabla v|_A^p dx - p \int_{\omega'} \left(\frac{\varphi_k}{v} \right)^{p-1} \nabla \varphi_k \cdot A(x) \nabla v |\nabla v|_A^{p-2} dx.$$

Recall that $L_A(\varphi_k, v) \geq 0$. Thus, by the Cauchy-Schwarz inequality we obtain

$$\int_{\omega'} |\nabla \varphi_k|_A^p dx \leq \int_{\Omega} L_A(\varphi_k, v) dx + p \int_{\omega'} |\nabla \varphi_k|_A \left(\frac{\varphi_k}{v} |\nabla v|_A \right)^{p-1} dx - (p-1) \int_{\omega'} \left(\frac{\varphi_k}{v} \right)^p |\nabla v|_A^p dx.$$

Young's inequality, $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$, with $a := (1-\varepsilon)^{1/p} |\nabla \varphi_k|_A$, $b := (1-\varepsilon)^{-1/p} \left(\frac{\varphi_k}{v} |\nabla v|_A \right)^{p-1}$ implies

$$\varepsilon \int_{\omega'} |\nabla \varphi_k|_A^p dx \leq \int_{\Omega} L_A(\varphi_k, v) dx + (p-1) \left(\frac{1}{(1-\varepsilon)^{1/(p-1)}} - 1 \right) \int_{\omega'} |\nabla(\log v)|_A^p \varphi_k^p dx.$$

By Proposition 3.3, $\int_{\Omega} L_A(\varphi_k, v) dx = o(1)$, where $o(1) \rightarrow 0$ as $k \rightarrow \infty$. Hence,

$$\int_{\omega'} |\nabla \varphi_k|_A^p dx \leq o(1) + C(\omega) \int_{\omega'} \varphi_k^p dx. \quad (6.4)$$

Step 2. Set: $\omega_0 := \left\{ x \in \omega \mid \exists \rho(x) \in (0, d(x, \Omega \setminus \omega)) \text{, } \sup_k \int_{B_{\rho(x)}(x)} |\varphi_k|^p dx < \infty \right\}$.

Since $\int_B |\varphi_k|^p dx = 1$, we get that $B \subset \omega_0$. Moreover, ω_0 is readily an open set. We claim that ω_0 is relatively closed in ω . Indeed, we use the following version of Poincaré inequality [12, Theorem 8.11]. For fix $0 < r < 1$ we have

$$\int_{B_1(0)} \left| u - \frac{\int_{B_r(0)} u dy}{\omega_n r^n} \right|^p dx \leq C(p, n, r) \int_{B_1(0)} |\nabla u|^p dx \quad \forall u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n).$$

On the other hand, there exists $\theta := \theta_B > 0$ such that $|\nabla u|^p \leq \theta^{-\frac{p}{2}} |\nabla u|_A^p$. Hence, in light of the inequality $2^{1-p} |a|^p - |b|^p \leq |a - b|^p$, we get

$$2^{1-p} \int_{B_1(0)} |u|^p dx - \int_{B_1(0)} \left| \frac{\int_{B_r(0)} u dy}{\omega_n r^n} \right|^p dx \leq \theta^{-\frac{p}{2}} C(p, n, r) \int_{B_1(0)} |\nabla u|_A^p dx.$$

Hence, for all $u \in W^{1,p}(\mathbb{R}^n)$ we obtain

$$\int_{B_1(0)} |u|^p dx \leq \tilde{C} \int_{B_1(0)} |\nabla u|^p dx + \frac{C}{r^n} \int_{B_r(0)} |u|^p dx. \quad (6.5)$$

By scaling, we obtain for $0 < a < 1$ that

$$\int_{B_a(0)} |u|^p dx \leq \tilde{C} a^p \int_{B_a(0)} |\nabla u|^p dx + C r^{-n} \int_{B_{ar}(0)} |u|^p dx \quad \forall u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n).$$

Hence, for every $\varepsilon > 0$ and $\rho > 0$ small enough, take $a_\varepsilon < \min \left\{ \left(\frac{\varepsilon}{\tilde{C}} \right)^{1/p}, \frac{\rho}{r} \right\}$, so that

$$\int_{B_{a_\varepsilon}(0)} |u|^p dx \leq \varepsilon \int_{B_{a_\varepsilon}(0)} |\nabla u|_A^p dx + C(\varepsilon, \rho) \int_{B_\rho(0)} |u|^p dx. \quad (6.6)$$

Therefore, we get for every $x \in \omega$, $\varepsilon > 0$, $\delta \in (0, \min \{a_\varepsilon, d(x, \partial\omega)\})$, and $u \in W_0^{1,p}(\Omega)$, that the following inequality holds

$$\int_{B_\delta(x)} |u|^p dx \leq \varepsilon \int_{B_\delta(x)} |\nabla u|_A^p dx + C(\varepsilon, \rho) \int_{B_\rho(x)} |u|^p dx. \quad (6.7)$$

Let $x_j \in \omega_0$, $x_j \rightarrow x_0 \in \omega$. Let $\varepsilon < \frac{1}{2C(\omega)}$, where $C(\omega)$ is the constant in (6.4). Let a_ε be as in (6.6) and fix $\delta \in (0, \min \{a_\varepsilon, d(x, \partial\omega)\})$, and $u \in W_0^{1,p}(\Omega)$. Finally, fix j such that $|x_0 - x_j| < \frac{\delta}{2}$. Then, with $\rho = \rho(x_j)$, we get from (6.4) and (6.7) that

$$\frac{1}{2C(\omega)} \int_{B_\delta(x_j)} |\nabla \varphi_k|_A^p dx \leq C(\varepsilon, \rho(x_j)) \int_{B_{\rho(x_j)}(x_j)} |\varphi_k|^p dx + o(1), \quad (6.8)$$

where $o(1) \rightarrow 0$ as $k \rightarrow \infty$. Thus $\int_{B_\delta(x_j)} |\nabla \varphi_k|_A^p dx$ is bounded (in k), and by (6.7), $\int_{B_\delta(x_j)} |\varphi_k|^p dx$ is also bounded. By the choice of j , we have that $B_{\frac{\delta}{2}}(x_0) \subset B_\delta(x_j)$, it follows that $\int_{B_{\frac{\delta}{2}}(x_0)} |\varphi_k|^p dx$ is bounded and consequently, $x_0 \in \omega_0$. So, ω_0 is also relatively closed in ω . Since ω is connected, we obtain $\omega_0 = \omega$. Hence, $\{\varphi_k\}$ is bounded in $L_{\text{loc}}^p(\Omega)$, and by (6.4) it follows that $\{\varphi_k\}$ is bounded in $W_{\text{loc}}^{1,p}(\Omega)$.

Step 3. We may assume (up to a subsequence) that $\varphi_k \rightharpoonup u$ in $W_{\text{loc}}^{1,p}(\Omega)$, and $\varphi_k \rightarrow u$ in $L_{\text{loc}}^p(\Omega)$. Let $\omega \Subset \Omega$ be a smooth domain, and denote

$$\mathcal{Q}_A^\omega(u) := \int_\omega L_A(u, v) dx = \int_\omega \mathcal{L}_A(x, u, \nabla u) dx \quad u \in W^{1,p}(\omega).$$

Claim: $\mathcal{Q}_A^\omega(u)$ is weakly lower semicontinuous in $W^{1,p}(\omega)$. Indeed, the sum of the first two terms of the Lagrangian $\mathcal{L}_A(x, z, q)$ is equal to $|q|_A^p + (p-1) \frac{|\nabla v|_A^p}{v^p} |z|^p$ which is convex in q . Therefore, the corresponding functional is weakly lower semicontinuous in $W^{1,p}(\omega)$. So, it remains to prove that the functional

$$\mathcal{J}_A^\omega(u) := \int_\omega \frac{u^{p-1}}{v^{p-1}} \nabla u \cdot A(x) \nabla v |\nabla v|_A^{p-2} dx \quad u \in W^{1,p}(\omega) \quad (6.9)$$

is weakly continuous on any sequence $\{\varphi_k\}$ satisfying $\varphi_k \rightharpoonup u$ in $W^{1,p}(\omega)$. Indeed

$$\begin{aligned} \mathcal{J}_A^\omega(\varphi_k) - \mathcal{J}_A^\omega(u) &= \int_\omega \frac{|\nabla v|_A^{p-2}}{v^{p-1}} \nabla v \cdot A(x) \nabla \varphi_k (\varphi_k^{p-1} - u^{p-1}) \, dx + \\ &\quad \int_\omega \nabla(\varphi_k - u) \cdot A(x) \nabla v |\nabla v|_A^{p-2} \frac{u^{p-1}}{v^{p-1}} \, dx. \end{aligned} \quad (6.10)$$

For a renamed subsequence, there exists a $U \in L^p(\omega)$, such that $0 \leq \varphi_k \leq U$ and $\varphi_k \rightarrow u$ a.e. in ω . Therefore $\varphi_k^{p-1} \leq U^{p-1} \in L^{p'}(\omega)$. Since φ_k, u, U are nonnegative we have that

$$|\varphi_k^{p-1} - u^{p-1}|^{p'} \leq (U^{p-1} + u^{p-1})^{p'} \leq C |U^p + u^p| \in L^1(\omega). \quad (6.11)$$

Hence by Hölder's inequality and Lebesgue's dominated convergence theorem,

$$\left| \int_\omega \frac{1}{v^{p-1}} |\nabla v|_A^{p-2} \nabla v A(x) \nabla \varphi_k (\varphi_k^{p-1} - u^{p-1}) \, dx \right| \leq C \|\nabla \varphi_k\|_p \|\varphi_k^{p-1} - u^{p-1}\|_{p'} \rightarrow 0. \quad (6.12)$$

Consider the functional

$$\Phi(w) := \int_\omega \nabla w \cdot A(x) \nabla v |\nabla v|_A^{p-2} \frac{u^{p-1}}{v^{p-1}} \, dx \quad w \in W^{1,p}(\omega).$$

Note $A(x) \nabla v |\nabla v|_A^{p-2} v^{1-p} u^{p-1} \in L^{p'}(\omega)$. Therefore, Hölder's inequality implies that Φ is a continuous functional on $W^{1,p}(\omega)$. Hence, by the definition of weak convergence, the second term of the right hand side of (6.10) converges to zero.

Therefore, $\mathcal{Q}_A^\omega(u)$ is weakly lower semicontinuous, and we conclude that $0 \leq \mathcal{Q}_A^\omega(u) \leq \liminf_{k \rightarrow \infty} \mathcal{Q}_A^\omega(\varphi_k) = 0$. Moreover, $\int_B u^p \, dx = 1$ and u is a ground state. Since $\mathcal{Q}_A^\omega(u) = 0$ for every subdomain $\omega \Subset \Omega$ containing B , it follows that $L_A(u, v) = 0$, and by Proposition 3.1 $u = cv$, where $c^{-1} = (\int_B v^p \, dx)^{1/p}$. Hence, any null sequence converges to the same positive supersolution v , and this implies the uniqueness claim. \square

We prove now Theorem 6.1.

Proof of Theorem 6.1. Lemma 6.3 implies part (1) and the necessity parts of (2) and (3). On the other hand if $\mathcal{Q}_{A,V}$ admits a null sequence, then by Lemma 6.3, (1.2) admits a unique positive continuous supersolution. The latter implies that the inequality $\mathcal{Q}_{A,V} \geq 0$ cannot be improved, Consequently, $\mathcal{Q}_{A,V}$ is critical in Ω , and hence null sequences exist.

To complete the proof of part (2), we need to show that if $\mathcal{Q}_{A,V}$ is critical in Ω , then it admits also a null sequence that converges locally uniformly in Ω . Let $\{\Omega_N\}_{N=1}^\infty$ be an exhaustion of Ω such that $x_0 \in \Omega_1$. Pick a nonzero nonnegative function $W \in C_0^\infty(\Omega_1)$. For $t \geq 0$ and $N \geq 1$, consider the functional $\mathcal{Q}_{A,V-tW}$ on $C_0^\infty(\Omega_N)$. By Proposition 7.4, there exists a unique $t_N > 0$ such that the functional $\mathcal{Q}_{A,V-t_N W}$ is critical in Ω_N (note the proof of Proposition 7.4 relies only on lemmas 6.2 and 6.3 but not on our theorem). Denote by ϕ_N the corresponding ground state satisfying $\phi_N(x_0) = 1$. Clearly, $\{t_N\}$ is a positive nonincreasing sequence that converges to $t_\infty \geq 0$. By Harnack's convergence principle, up to a subsequence, $\{\phi_N\}$ converges locally uniformly to a positive solution v of the equation $\mathcal{Q}_{A,V-t_\infty W}(u) = 0$ in Ω .

On the other hand, since $\mathcal{Q}_{A,V}$ is critical in Ω , it follows that for any $t > 0$ the functional $\mathcal{Q}_{A,V-tW} \not\geq 0$ on $C_0^\infty(\Omega)$, hence $t_\infty = 0$. By the uniqueness of the positive (super)solutions of the equation $Q_{A,V}(u) = 0$ in Ω , it follows that the whole sequence converges locally uniformly to v . Since $\lambda_1(Q_{A,V-t_N W}, \Omega_N) = 0$, it follows that ϕ_N is a principal eigenfunction of $\mathcal{Q}_{A,V-t_N W}$ and therefore, $\phi_N \in W_0^{1,p}(\Omega_N) \subset W_0^{1,p}(\Omega)$. Consequently,

$$\mathcal{Q}_{A,V}(\phi_N) = t_N \int_{\Omega} W |\phi_N|^p dx \rightarrow 0, \quad \text{and} \quad \int_B |\phi_N|^p dx \asymp 1.$$

By Remark 2.8, $\{\phi_N\}$ is a null sequence of $\mathcal{Q}_{A,V}$ and v is the ground state of $\mathcal{Q}_{A,V}$.

In light of Lemma 6.3, part (4) is proved in Lemma 6.2. So, it remains to prove part (5). Consider the functional

$$\tilde{\mathcal{Q}}(\varphi) := \mathcal{Q}_{A,V}(\varphi) + \int_B |\varphi|^p dx \quad \varphi \in C_0^\infty(\Omega),$$

where $B \Subset \Omega$, is a fixed open set. Clearly, $\tilde{\mathcal{Q}}$ is subcritical in Ω . By part (4), we have that for any $B \Subset \Omega$, there exists a positive continuous function W in Ω such that

$$\int_{\Omega} W(x) |\varphi(x)|^p dx \leq \mathcal{Q}_{A,V}(\varphi) + \int_B |\varphi|^p dx \quad \forall \varphi \in C_0^\infty(\Omega). \quad (6.13)$$

Suppose now that for each open ball $B \Subset \Omega$ there is a nonnegative sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ such that $\int_B |\varphi_k|^p dx = 1$, $\mathcal{Q}_{A,V}(\varphi_k) \rightarrow 0$ and $\int_{\Omega} \varphi_k \psi dx \rightarrow 0$. Since $\{\varphi_k\}$ is a null sequence, by Lemma 6.3, it converges to a ground state $\phi > 0$ in $L_{\text{loc}}^p(\Omega)$. Then $\int_{\Omega} \varphi_k \psi dx \rightarrow \int_{\Omega} \phi \psi dx \neq 0$, and we arrive at a contradiction. Therefore, there exists an open $B \Subset \Omega$ such that

$$\int_B |\varphi|^p dx \leq C \left(\mathcal{Q}_{A,V}(\varphi) + \left| \int_{\Omega} \varphi \psi dx \right|^p \right) \quad \forall \varphi \in C_0^\infty(\Omega). \quad (6.14)$$

Hence, (6.13) and (6.14) imply (6.1). \square

As a consequence we generalize Proposition 2 in [1] concerning the principle eigenvalue.

Lemma 6.4. *Let Ω be a bounded domain, $A \in L^\infty(\Omega, \mathbb{R}^{n^2})$ symmetric and uniformly positive definite, $V \in L^\infty(\Omega)$, and $\mathcal{Q}_{A,V} \geq 0$ in Ω . If $p < 2$ assume further that $A \in C^\alpha(\Omega)$.*

1. *If (4.12) admits a real eigenvalue λ with an eigenfunction $\psi \geq 0$, then $\lambda = \lambda_1$.*
2. *λ_1 is a simple eigenvalue.*
3. *The principal eigenfunction associated with λ_1 is the unique positive supersolution of the equation $Q_{A,V-\lambda_1}(u) = 0$ in Ω .*
4. *If $\lambda \neq \lambda_1$ is a real eigenvalue with an eigenfunction ψ , then*

$$\text{meas}(\Omega^-) \geq (\|\lambda - V\|_\infty \bar{C}^p)^\sigma, \quad (6.15)$$

where $\Omega^- = \{x \in \Omega : \psi(x) < 0\}$, \bar{C} is a constant independent of ψ and λ , and $\sigma := -\frac{n}{p}$ if $p < n$, $\sigma := -2$ if $p \geq n$.

5. *λ_1 is an isolated eigenvalue in \mathbb{R} .*

Remark 6.5. If λ is an eigenvalue of (4.12) and $\Omega \in \mathbb{R}^n$, then $\|\lambda - V\|_\infty > 0$. Otherwise, an associated eigenfunction u is p -harmonic in Ω and vanishes on $\partial\Omega$. Hence, $u = 0$ in Ω .

Proof of Lemma 6.4. 1–3. Any eigenfunction with eigenvalue λ_1 is a minimizer of (4.13). Lemma 4.9 implies that (4.13) admits a minimizer ϕ , and that each minimizer is a principal eigenfunction of the operator $Q_{A,V}$ with eigenvalue λ_1 (up to a sign change). Moreover, if ψ is a principal eigenfunction with eigenvalue λ , then ψ is a positive supersolution of the equation $Q_{A,V-\lambda_1}(u) = 0$ in Ω . Since any minimizing sequence is a null sequence (up to a sign change), part (3) of Theorem 6.1 implies that $\lambda = \lambda_1$, that λ_1 is simple, and that ϕ is the unique positive supersolution of the equation $Q_{A,V-\lambda_1}(u) = 0$ in Ω .

4. Recall that $|\psi_-| \in W_0^{1,p}(\Omega)$, and by Lemma 2.4, $|\psi_-|$ is a subsolution of the equation $Q_{A,V-\lambda}(u) = 0$ in Ω . Hence,

$$\|\nabla \psi_-\|_{p,A}^p \leq \int_\Omega (\lambda - V(x)) |\psi_-|^p dx \leq \|\lambda - V\|_\infty \int_\Omega |\psi_-|^p dx.$$

Hölder inequality implies that

$$\|\nabla \psi_-\|_{p,A}^p \leq \|\lambda - V\|_\infty \|\psi_-\|_{p^*}^p (\text{meas}(\Omega_-))^{1-\frac{p}{p^*}}, \quad (6.16)$$

where $p^* = \frac{pn}{n-p}$ if $p < n$, and $p^* = 2p$ if $p \geq n$. By Gagliardo-Nirenberg-Sobolev inequality,

$$\|\psi_-\|_{p^*} \leq C \|\nabla \psi_-\|_p \leq \bar{C} \|\nabla \psi_-\|_{p,A}.$$

Therefore,

$$\|\nabla \psi_-\|_{p,A}^p \leq \|\lambda - V\|_\infty \bar{C}^p \|\nabla \psi_-\|_{p,A}^p (\text{meas}(\Omega_-))^{1-\frac{p}{p^*}},$$

and we obtain the desired result.

5. Assume that there exists a sequence of eigenvalues $\{\lambda_k\} \subset \mathbb{R}$ such that $\lambda_k > \lambda_1$, $\lim_{k \rightarrow \infty} \lambda_k = \lambda_1$. Let $\{\phi_k\} \in W_0^{1,p}(\Omega)$ be the corresponding normalized eigenfunctions. By compactness and the simplicity of λ_1 , $\lim_{k \rightarrow \infty} \phi_k = \phi$ in $W_0^{1,p}(\Omega)$, where $\phi > 0$ is the principal eigenfunction with eigenvalue λ_1 . By part (4), and Remark 6.5, $\text{meas}\{\phi_k < 0\} \geq c > 0$, where c is independent of k , a contradiction to the positivity of ϕ . \square

Corollary 6.6. *Assume that Ω is a bounded domain, A is a bounded measurable symmetric matrix which is uniformly positive definite in Ω , and $V \in L^\infty(\Omega)$. If $p < 2$ assume further that $A \in C^\alpha(\Omega)$. Then all the assertions of Theorem 4.10 are equivalent.*

Proof. (iv') \Rightarrow (iii). By the AAP theorem $\lambda_1 \geq 0$. Lemma 6.4 implies that the principal eigenfunction is the unique positive supersolution of the equation $Q_{A,V-\lambda_1}(u) = 0$ in Ω . Hence, $\lambda_1 > 0$ and (iv') \Rightarrow (iii) (cf. the proof of this part in [9], where the WCP was used under the assumption $\lambda_1 \leq 0$). Thus, all the assertions of Theorem 4.10 are equivalent. \square

As in [20], criticality is characterized in terms of the relative capacity of compact sets.

Definition 6.7. Suppose that the functional $\mathcal{Q}_{A,V}$ is nonnegative on $C_0^\infty(\Omega)$. Let $K \Subset \Omega$ be a compact set. The $\mathcal{Q}_{A,V}$ -capacity of K in Ω is defined by

$$\text{Cap}_{\mathcal{Q}_{A,V}}(K, \Omega) := \inf \{ \mathcal{Q}_{A,V}(\varphi) \mid \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \text{ on } K \}.$$

The following theorem generalizes Theorem 3.4 in [19] and Theorem 4.5 in [20]. We omit the proof since it differs only slightly from the proofs in [19, 20].

Theorem 6.8. *Suppose that $\mathcal{Q}_{A,V} \geq 0$ in Ω , where $A \in C^\alpha(\Omega, \mathbb{R}^{n^2})$ satisfy conditions (A) and (E), and $V \in L^\infty_{\text{loc}}(\Omega)$. Then following statement are equivalent.*

(i) $\mathcal{Q}_{A,V}$ is subcritical in Ω .

(ii) There exists a continuous function $W > 0$ in Ω such that

$$\mathcal{Q}_{A,V}(\varphi) \geq \int_{\Omega} W(x)(|\nabla \varphi|_A^p + |\varphi|^p) dx \quad \forall \varphi \in C_0^\infty(\Omega). \quad (6.17)$$

(iii) There exists an open set $B \Subset \Omega$ and $c_B > 0$ such that

$$\mathcal{Q}_{A,V}(\varphi) \geq c_B \left| \int_B |\varphi| dx \right|^p \quad \forall \varphi \in C_0^\infty(\Omega). \quad (6.18)$$

(iv) The $\mathcal{Q}_{A,V}$ -capacity of any closed ball in Ω is positive.

Suppose further that $p < n$ and let $p^* := \frac{pn}{n-p}$ be its critical Sobolev exponent. Then $\mathcal{Q}_{A,V}$ is subcritical in Ω if and only if the following Hardy-Sobolev-Maz'ya-type inequality holds true: there exists a continuous function $\tilde{W} > 0$ in Ω such that

$$\mathcal{Q}_{A,V}(\varphi) \geq \left(\int_{\Omega} \tilde{W}(x) |\varphi|^{p^*} dx \right)^{\frac{p}{p^*}} \quad \forall \varphi \in C_0^\infty(\Omega). \quad (6.19)$$

7. Criticality Theory

In this short section we list further positivity properties of $\mathcal{Q}_{A,V}$, generalizing the results of [18, Section 4]. We omit the proofs since they differ only slightly from the proofs in [18].

Proposition 7.1. *Let $V_1, V_2 \in L^\infty_{\text{loc}}(\Omega)$ and suppose that $V_2 \not\geq V_1$.*

1. *If $\mathcal{Q}_{A,V_1} \geq 0$ on $C_0^\infty(\Omega)$, then \mathcal{Q}_{A,V_2} is subcritical in Ω .*
2. *If \mathcal{Q}_{A,V_2} is critical in Ω , then \mathcal{Q}_{A,V_1} is supercritical in Ω .*

Proposition 7.2. *Let $\Omega_1 \subset \Omega_2$ be domains in \mathbb{R}^n such that $\Omega_2 \setminus \bar{\Omega}_1 \neq \emptyset$.*

1. *If $\mathcal{Q}_{A,V} \geq 0$ on $C_0^\infty(\Omega_2)$, then $\mathcal{Q}_{A,V}$ is subcritical in Ω_1 .*
2. *If $\mathcal{Q}_{A,V}$ is critical in Ω_1 , then $\mathcal{Q}_{A,V}$ is supercritical in Ω_2 .*

Proposition 7.3. *Let $V_0, V_1 \in L^\infty_{\text{loc}}(\Omega)$. For $t \in \mathbb{R}$ we denote*

$$\mathcal{Q}_{A,t}(\varphi) := (1-t)\mathcal{Q}_{A,V_0}(\varphi) + t\mathcal{Q}_{A,V_1}(\varphi) \quad \varphi \in C_0^\infty(\Omega). \quad (7.1)$$

Suppose that $\mathcal{Q}_{A,V_i} \geq 0$ on $C_0^\infty(\Omega)$ for $i = 0, 1$. Then the functional $\mathcal{Q}_{A,t} \geq 0$ on $C_0^\infty(\Omega)$ for all $t \in [0, 1]$. Moreover, if $V_0 \neq V_1$, then $\mathcal{Q}_{A,t}$ is subcritical in Ω for all $t \in (0, 1)$.

Proposition 7.4. *Let $\mathcal{Q}_{A,V}$ be a subcritical functional in Ω . Consider $V_0 \in L^\infty(\Omega)$ such that $V_0 \not\geq 0$ and $\text{supp } V_0 \Subset \Omega$. Then there exist $\tau_+ > 0$ and $-\infty \leq \tau_- < 0$ such that $\mathcal{Q}_{A,V+tV_0}$ is subcritical in Ω if and only if $t \in (\tau_-, \tau_+)$, and $\mathcal{Q}_{A,V+\tau_+V_0}$ is critical in Ω .*

Proposition 7.5. *Let $\mathcal{Q}_{A,V}$ be a critical functional in Ω , and let ϕ be the corresponding ground state, Consider $V_0 \in L^\infty(\Omega)$ such that $\text{supp } V_0 \Subset \Omega$. Then there exists $0 < \tau_+ \leq \infty$ such that $\mathcal{Q}_{A,V+tV_0}$ is subcritical in Ω for $t \in (0, \tau_+)$ if and only if $\int_{\Omega} V_0 |\phi|^p dx > 0$.*

8. Liouville Theorem

In the present section we generalize Liouville-comparison theorems proved in [15, 17] (see also references therein). We note that the results in [17] are the counterpart of the results in [15] proved for the linear case ($p = 2$) for operators of the form $Pu := -\nabla \cdot (A(x)\nabla u) + V(x)u$ with a measurable symmetric matrix valued function A which is locally bounded and locally uniformly positive definite, while [17] deals with p -Laplace type equation. We have:

Theorem 8.1. *For $j = 0, 1$, consider the functional*

$$\mathcal{Q}_{A_j, V_j}(\varphi) := \int_{\Omega} \left(|\nabla \varphi|_{A_j}^p + V_j(x)|\varphi|^p \right) dx \quad \varphi \in C_0^\infty(\Omega), \quad (8.1)$$

where the matrix A_j and the potential V_j satisfy assumptions (A), (E), and (V). If $p < 2$ assume further that $A_j \in C^\alpha(\Omega)$. Suppose that the following assumptions hold true:

- (i) The functional \mathcal{Q}_{A_1, V_1} admits a ground state ϕ in Ω .
- (ii) $\mathcal{Q}_{A_0, V_0} \geq 0$ in Ω , and the equation $\mathcal{Q}_{A_0, V_0}(u) = 0$ in Ω admits a subsolution $\psi \in W_{\text{loc}}^{1,p}(\Omega)$ satisfying $\psi_+ := \max\{0, \psi\} \neq 0$.
- (iii) There exists $M > 0$ such that

$$\psi_+^2 A_0 \leq M^2 \phi^2 A_1 \quad \text{a. e. in } \Omega. \quad (8.2)$$

That is, for almost all $x \in \Omega$ the matrix $(M\phi(x))^2 A_1(x) - (\psi_+(x))^2 A_0(x)$ is non negative-definite on \mathbb{R}^n .

- (iv) There exists $N > 0$ such that

$$|\nabla \psi|_{A_0}^{p-2} \leq N^{p-2} |\nabla \phi|_{A_1}^{p-2} \quad \text{a. e. in } \Omega \cap \{\psi > 0\}. \quad (8.3)$$

Then the functional \mathcal{Q}_{A_0, V_0} is critical in Ω , and ψ is the unique positive supersolution of the equation $\mathcal{Q}_{A_0, V_0}(u) = 0$ in Ω .

Proof of Theorem 8.1. By Lemma 2.4, we may assume that $\psi \geq 0$.

Let $\{\varphi_k\} \subset C_0^\infty(\Omega)$ be a null sequence for \mathcal{Q}_{A_1, V_1} . So, there exists an open set $B \Subset \Omega$ such that $\int_B |\varphi_k|^p dx = 1$, and $\lim_{k \rightarrow \infty} \mathcal{Q}_{A_1, V_1}(\varphi_k) = 0$. Without loss of generality, we may assume that $B \subset \text{supp } \psi$. Let $w_k := \frac{\varphi_k}{\phi} \geq 0$. From (3.9) it follows that there exists $C_1 > 0$ such that

$$\int_{\Omega} \phi^2 |\nabla w_k|_{A_1}^2 \left(w_k |\nabla \phi|_{A_1} + \phi |\nabla w_k|_{A_1} \right)^{p-2} dx \leq C_1 \mathcal{Q}_{A_1, V_1}(\varphi_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (8.4)$$

Fix $\alpha \in \mathbb{R}_+$, and consider the function $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ defined by

$$f(s, t) := t^2 \left(\alpha s^{\frac{1}{p-2}} + t \right)^{p-2}.$$

Since

$$f(s, t) = \alpha t^2 s^{\frac{3-p}{p-2}} \left(\alpha s^{\frac{1}{p-2}} + t \right)^{p-3} \geq 0, \quad f_t(s, t) = t \left(\alpha s^{\frac{1}{p-2}} + t \right)^{p-3} \left(2\alpha s^{\frac{1}{p-2}} + tp \right) \geq 0,$$

the function $f(s, t)$ is a nondecreasing monotone function in each variable separately. Set $t_0 := \psi|\nabla w_k|_{A_0}$ and $t_1 := M\phi|\nabla w_k|_{A_1}$, then assumptions (8.2) implies that $t_0 \leq t_1$. Hence,

$$I_k := \int_{\Omega} \psi^2 |\nabla w_k|_{A_0}^2 \left(w_k |\nabla \psi|_{A_0} + \psi |\nabla w_k|_{A_0} \right)^{p-2} dx \leq M^2 \int_{\Omega} \phi^2 |\nabla w_k|_{A_1}^2 \left(w_k |\nabla \psi|_{A_1} + M\phi |\nabla w_k|_{A_1} \right)^{p-2} dx.$$

Let now $s_0 := |\nabla \psi|_{A_0}^{p-2}$ and $s_1 := N^{p-2} |\nabla \phi|_{A_1}^{p-2}$, then (8.3) implies that $s_0 \leq s_1$. Therefore,

$$\begin{aligned} I_k &\leq M^2 \int_{\Omega} \phi^2 |\nabla w_k|_{A_1}^2 \left(N w_k |\nabla \phi|_{A_1} + M\phi |\nabla w_k|_{A_1} \right)^{p-2} dx \leq \\ &\quad C_2 \int_{\Omega} \phi^2 |\nabla w_k|_{A_1}^2 \left(w_k |\nabla \phi|_{A_1} + \phi |\nabla w_k|_{A_1} \right)^{p-2} dx \leq C_2 C_1 \mathcal{Q}_{A_1, V_1}(\varphi_k) \rightarrow 0. \end{aligned}$$

By (3.10) we have $\mathcal{Q}_{A_0, V_0}(\psi w_k) \leq C_3 I_k$. So, $\mathcal{Q}_{A_0, V_0}(\psi w_k) \rightarrow 0$. Since $w_k \rightarrow 1$ in $L_{\text{loc}}^p(\Omega)$, it follows that $\psi w_k \rightarrow \psi$ in $L_{\text{loc}}^p(\Omega)$. Consequently, $\int_B \phi^p w_k^p dx = 1$ implies that $\int_B \psi^p w_k^p dx \asymp 1$. In light of Remark 2.8, we conclude that ψ is a ground state of \mathcal{Q}_{A_0, V_0} . \square

Example 8.2 (cf. [17] examples 3.1–3.3). Theorem 8.1 implies that we may replace the p -Laplacian appearing as the principal part of Q_0 in examples 3.1–3.3 of [17] with the (p, A_0) -Laplacian, where A_0 is a *bounded* measurable symmetric matrix which is locally uniformly positive definite in \mathbb{R}^n .

Example 8.3. Let $1 \leq n \leq p$, $p > 1$, $\Omega = \mathbb{R}^n$, and consider the functional

$$\mathcal{Q}_{A_1, V_1}(\varphi) := \mathcal{Q}_{I, 0}(\varphi) = \int_{\mathbb{R}^n} |\nabla \varphi|^p dx \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

By Example 2.9 the functional \mathcal{Q}_{A_1, V_1} admits a ground state $\phi = \text{const.}$ in \mathbb{R}^n .

Consider the functional

$$\mathcal{Q}_{A_0, V_0}(\varphi) := \mathcal{Q}_{A_0, 0}(\varphi) = \int_{\Omega} |\nabla \varphi|_{A_0}^p dx \quad \varphi \in C_0^\infty(\mathbb{R}^n),$$

where A_0 is a *bounded* measurable symmetric matrix which is locally uniformly positive definite in \mathbb{R}^n . Then $\psi = \text{const.} > 0$ is a positive (sub)solution of the equation $Q_{A_0, V_0}(u) = 0$ in \mathbb{R}^n . By Theorem 8.1, ψ is the unique positive supersolution and unique bounded solution of the equation $Q_{A_0, V_0}(u) = 0$ in \mathbb{R}^n (cf. 6.10 and 6.11 of [10]).

9. Minimal Growth

In this section we generalize some results in [20, Section 5] concerning the existence of positive solutions of minimal growth in a neighborhood of infinity in Ω .

Definition 9.1. Let K_0 be a compact set in Ω . A positive solution u of the equation $Q_{A, V}(w) = 0$ in $\Omega \setminus K_0$ is said to be a *positive solution of minimal growth in a neighborhood of infinity in Ω* (or $u \in \mathcal{M}_{\Omega, K_0}$ for brevity) if for any compact set K in Ω , with a smooth boundary, such that $K_0 \Subset \text{int}(K)$, and any positive supersolution $v \in C((\Omega \setminus K) \cup \partial K)$ of the equation $Q_{A, V}(w) = 0$ in $\Omega \setminus K$, the inequality $u \leq v$ on ∂K implies that $u \leq v$ in $\Omega \setminus K$.

If $u \in \mathcal{M}_{\Omega, \emptyset}$, then u is called a *global minimal solution of the equation $Q_{A, V}(w) = 0$ in Ω* .

Remark 9.2. Suppose that $\mathcal{Q}_{A,V} \geq 0$ in Ω , and let $v > 0$ be a positive solution of (1.2). Lemma 5.1 implies that $\lambda_1(\tilde{\Omega}) > 0$ for any bounded domain $\tilde{\Omega} \Subset \Omega$.

Let $\{\Omega_N\}_{N=1}^\infty$ be an exhaustion of Ω . Fix $K \Subset \Omega$ with smooth boundary, and let $u \in C^\alpha(\partial K)$ be a positive function. Let u_N be a solution of the following Dirichlet problem

$$\begin{cases} Q_{A,V}(w) = 0 & \text{in } \Omega_N \setminus K, \\ w = u & \text{on } \partial K, \\ w = 0 & \text{on } \partial\Omega_N. \end{cases} \quad (9.1)$$

By Theorem 4.10, problem (9.1) is solvable. Moreover, by the WCP (Theorem 4.13), $\{u_N\}$ is a monotone nondecreasing sequence, satisfying $u_N \leq cv$ in $\Omega_N \setminus K$, where $c := \|\frac{u(x)}{v(x)}\|_{\infty, K}$. Denote: $u^K := \lim_{N \rightarrow \infty} u_N$ on $\Omega \setminus K$.

Clearly, $u^K \leq cv$. Moreover, a comparison argument implies that u^K does not depend on the exhaustion $\{\Omega_N\}_{N=1}^\infty$. Moreover, by the Harnack convergence principle, u^K is a positive solution of the equation $Q_{A,V}(w) = 0$ in $\Omega \setminus K$.

A further comparison argument shows that

Lemma 9.3. *Let $\mathcal{Q}_{A,V}$, K , v and u as above. Then $u^K \leq cv$, and $u^K \in \mathcal{M}_{\Omega, K}$.*

Lemma 9.4. *Let $K_0 \Subset \Omega$, and let u be a positive solution of the equation $Q_{A,V}(w) = 0$ in $\Omega \setminus K_0$. Then $u \in \mathcal{M}_{\Omega, K_0}$ if and only if for any compact set $K \Subset \Omega$ with smooth boundary, such that $K_0 \Subset \text{int}(K)$, we have $u = u^K$.*

Proof. Fix a smooth compact set $K \Subset \Omega$ such that $K_0 \Subset \text{int}(K)$, and let $v \in C((\Omega \setminus K) \cup \partial K)$ be a positive supersolution of the equation $Q_{A,V}(w) = 0$ in $\Omega \setminus K$, satisfying the inequality $u \leq v$ on ∂K . Then by a standard comparison principle, we conclude that $u^K = \lim_{N \rightarrow \infty} u_N \leq v$ in $\Omega \setminus K$. Hence, if $u = u^K$ in $\Omega \setminus K$, it follows that $u \leq v$ in $\Omega \setminus K$. Hence, $u \in \mathcal{M}_{\Omega, K_0}$.

On the other hand, if $u \in \mathcal{M}_{\Omega, K_0}$, then u is a positive solution of the equation $Q_{A,V}(w) = 0$ in $\Omega \setminus K_0$. Therefore, as above, $u^K \leq u$ in $\Omega \setminus K$. By definition, $u \leq u^K$ in $\Omega \setminus K$. Thus, $u = u^K$ in $\Omega \setminus K$. \square

Theorem 9.5. *Suppose that $1 < p < \infty$, and $\mathcal{Q}_{A,V} \geq 0$ in Ω . Then for any $x_0 \in \Omega$ the equation $Q_{A,V}(w) = 0$ admits a positive solution $u \in \mathcal{M}_{\Omega, \{x_0\}}$.*

Proof. Consider an exhaustion $\{\Omega_N\}_{N=1}^\infty$ of Ω such that $x_0 \in \Omega_1$. Let $\{f_N\}$ be a sequence of nonzero nonnegative smooth functions such that $f_N \in C_0^\infty(B_{2/N}(x_0) \setminus \overline{B_{1/N}(x_0)})$.

Denote $A_N := \Omega_N \setminus \overline{B_{1/N}(x_0)}$, and choose a fixed reference point $x_1 \in A_1$. Let v_N be a positive solution of the Dirichlet problem

$$\begin{cases} Q_{A,V}(w) = f_N & \text{in } A_N, \\ w = 0 & \text{on } \partial A_N. \end{cases} \quad (9.2)$$

Set $u_N(x) := v_N(x)/v_N(x_1)$. By the Harnack convergence principle, $\{u_N\}$ admits a subsequence which converges locally uniformly in $\Omega \setminus \{x_0\}$ to a positive solution u of the equation $Q_{A,V}(w) = 0$ in $\Omega \setminus \{x_0\}$.

Let $K \Subset \Omega$ be a compact set with a smooth boundary such that $x_0 \in \text{int}(K)$, and let $v \in C((\Omega \setminus K) \cup \partial K)$ be a positive supersolution of the equation $Q_{A,V}(w) = 0$ in $\Omega \setminus K$ satisfying inequality $u \leq v$ on ∂K .

Now, for $\delta > 0$ there exists N_K such that $\text{supp} f_N \subset B_{2/N}(x_0) \Subset K$ for $N \geq N_K$, and $u_N \leq (1 + \delta)v$ on $\partial(\Omega_N \setminus K)$. Using a comparison argument, and letting $N \rightarrow \infty$, and then $\delta \rightarrow 0$, we obtain that $u \leq v$ in $\Omega \setminus K$. Hence, $u \in \mathcal{M}_{\Omega, \{x_0\}}$. \square

Theorem 9.6. *Suppose that the matrix A and the potential V satisfy conditions (A), (E) and (V). If $p < 2$ assume further that $A \in C^\alpha(\Omega)$. Then $\mathcal{Q}_{A,V}$ is subcritical in Ω if and only if (1.2) does not admit a global minimal solution in Ω . In particular, ϕ is a ground state of (1.2) if and only if ϕ is a global minimal solution of (1.2).*

Proof. Necessity: Assume that there exists a global minimal solution $u > 0$ of the equation $Q_{A,V}(w) = 0$ in Ω , and suppose that $\mathcal{Q}_{A,V}$ is subcritical in Ω . Then there exists a positive supersolution v satisfying $Q_{A,V}(v) \geq 0$ in Ω .

Clearly, there exists $\varepsilon > 0$ such that $\varepsilon u \leq v$ in Ω . Define

$$\varepsilon_0 := \max \{ \varepsilon > 0 \mid \varepsilon u \leq v \text{ in } \Omega \}.$$

Evidently, $\varepsilon_0 u \not\leq v$ in Ω . Therefore, there exist $\delta_1, \delta_2 > 0$ and $x_1 \in \Omega$ such that

$$(1 + \delta_1)\varepsilon_0 u(x) \leq v(x) \quad x \in B_{\delta_2}(x_1).$$

Hence, by the definition a global minimal solution it follows that

$$(1 + \delta_1)\varepsilon_0 u(x) \leq v(x) \quad x \in \Omega \setminus B_{\delta_2}(x_1).$$

Consequently, $(1 + \delta_1)\varepsilon_0 u(x) \leq v(x)$ in Ω , which contradicts the definition of ε_0 . Thus, subcriticality implies the nonexistence of a global minimal solution in Ω .

Sufficiency: Consider an exhaustion $\{\Omega_N\}_{N=1}^\infty$ of Ω such that $x_0 \in \Omega_1$, and $x_1 \in \Omega \setminus \Omega_1$. Assume that $\mathcal{Q}_{A,V}$ is critical in Ω , and let ϕ be its (unique) ground state satisfying $\phi(x_1) = 1$. We need to prove that ϕ is a global minimal solution of the equation $Q_{A,V}(w) = 0$ in Ω .

Indeed, fix $i \in \mathbb{N}$, and let $f_i \in C_0^\infty(B_{1/i}(x_0))$ satisfy $0 \leq f_i(x) \leq 1$. For $N \geq 1$, let $u_{N,i}$ be a positive solution of the Dirichlet problem

$$\begin{cases} Q_{A,V}(w) = f_i & \text{in } \Omega_N, \\ w = 0 & \text{on } \partial\Omega_N. \end{cases} \quad (9.3)$$

By the WCP, $\{u_{N,i}\}_{N \geq 1}$ is a nondecreasing sequence. If $\{v_{N,i}(x_1)\}$ is bounded, then the sequence converges to v_i , where v_i satisfies $Q_{A,V}(v_i) = f_i \geq 0$ in Ω . Due to Theorem 6.1 (part (3)), this contradicts our criticality assumption. Therefore, $v_{N,i}(x_1) \rightarrow \infty$ as $N \rightarrow \infty$.

Denote $u_{N,i}(x) := \frac{v_{N,i}(x)}{v_{N,i}(x_1)}$. By Harnack convergence principle, we may extract a subsequence of $\{u_{N,i}\}$ that converges as $N \rightarrow \infty$ to a positive solution u_i of the equation $Q_{A,V}(w) = 0$ in Ω . By the uniqueness of the ground state, we have $u_i = \phi$.

Let $K \Subset \Omega$ be a smooth compact set, we may assume that $x_0 \in \text{int}(K)$. Let $v \in C(\Omega \setminus \text{int}(K))$ be a positive supersolution of the equation $Q_{A,V}(w) = 0$ in $\Omega \setminus K$ such

that the inequality $\phi \leq v$ holds on ∂K . Let $i \in \mathbb{N}$ be sufficiently large number such that $\text{supp} f_i \Subset K$. For any $\delta > 0$ there exists N_δ such that for $N \geq N_\delta$ we have

$$\begin{cases} 0 = Q_{A,V}(u_{N,i}) \leq Q_{A,V}(v) & \text{in } \Omega_N \setminus K, \\ Q_{A,V}(v) \geq 0 & \text{in } \Omega_N \setminus K, \\ 0 \leq u_{N,i} \leq (1 + \delta)v & \text{on } \partial(\Omega_N \setminus K), \end{cases}$$

which implies that $\phi = u_i \leq (1 + \delta)v$ in $\Omega \setminus K$. Letting $\delta \rightarrow 0$ we obtain $\phi \leq v$ in $\Omega \setminus K$. Since $K \Subset \Omega$ is an arbitrary smooth compact set, it follows that the ground state ϕ is a global minimal solution of the equation $Q_{A,V}(w) = 0$ in Ω . \square

Suppose that u is a positive solution of the equation $Q_{A,V}(w) = 0$ in a punctured neighborhood of x_0 , and $1 < p \leq n$. Then by [23, 24], either u has a removable singularity, or

$$u(x) \asymp \begin{cases} |x - x_0|^{\alpha(n,p)} & p < n, \\ -\log|x - x_0| & p = n, \end{cases} \quad \text{as } x \rightarrow x_0, \quad (9.4)$$

where $\alpha(n, p) := (p - n)/(p - 1)$. In particular, in the nonremovable case $\lim_{x \rightarrow x_0} u(x) = \infty$.

Consequently, we have

Theorem 9.7. *Suppose that the matrix A and the potential V satisfy conditions (A), (E) and (V). If $p < 2$ assume further that $A \in C^\alpha(\Omega)$. Let $x_0 \in \Omega$, and let $u \in \mathcal{M}_{\Omega, \{x_0\}}$. Suppose that $\mathcal{Q}_{A,V}$ is subcritical in Ω , then u has a nonremovable singularity at x_0 .*

Assume that $1 < p \leq n$, $x_0 \in \Omega$, and $u \in \mathcal{M}_{\Omega, \{x_0\}}$. Suppose that u has a nonremovable singularity at x_0 , then $\mathcal{Q}_{A,V}$ is subcritical in Ω .

Proof. Let $u \in \mathcal{M}_{\Omega, \{x_0\}}$. If u has a removable singularity at x_0 , then its continuous extension \bar{u} is a global minimal solution in Ω . Hence, by Theorem 9.6, $\mathcal{Q}_{A,V}$ is critical.

Assume that $1 < p \leq n$. Let $u \in \mathcal{M}_{\Omega, \{x_0\}}$ and suppose that u has a nonremovable singularity at x_0 . If $\mathcal{Q}_{A,V}$ is critical in Ω , then by Theorem 9.6, there exists a global minimal solution v in Ω . Let $\varepsilon > 0$. Since $\lim_{x \rightarrow x_0} u(x) = \infty$, a comparison argument implies that $v \leq \varepsilon u$ in Ω . Hence, $v = 0$ which is a contradiction. Therefore, $\mathcal{Q}_{A,V}$ is subcritical in Ω . \square

Definition 9.8. A function $u \in \mathcal{M}_{\Omega, \{x_0\}}$ having a nonremovable singularity at x_0 is called a *minimal positive Green function* of $Q_{A,V}$ in Ω . We denote such a function by $G_{A,V}^\Omega(x, x_0)$.

Problems 9.9. 1. Prove the uniqueness of the positive minimal Green function $G_{A,V}^\Omega(x, x_0)$ and study its *asymptotic* behavior as $x \rightarrow x_0$.

2. Assume that $p > n$, and consider a nonnegative functional $\mathcal{Q}_{A,V}$. Is it true that $\mathcal{Q}_{A,V}$ is subcritical in Ω if $Q_{A,V}$ admits a positive minimal Green function $G_{A,V}^\Omega(x, x_0)$?

Note that [7, 20] give affirmative answers to the above problems for the case $A = I$.

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